Singularities of the Isospectral Hilbert Scheme

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Abstract

We study the singularities of the isospectral Hilbert scheme B^n of n points over a smooth algebraic surface and we prove that they are canonical if $n \leq 5$, log-canonical if $n \leq 7$ and not log-canonical if $n \geq 9$. We describe as well two explicit log-resolutions of B^3, one crepant and the other $\mathfrak{S}_3$-equivariant.

Introduction

The aim of this work is the study of the singularities of the isospectral Hilbert scheme of n points over a smooth complex algebraic surface. The isospectral Hilbert scheme B^n can be defined as the blow-up of the product variety X^n along the big diagonal Δ_n. The isospectral Hilbert scheme has been introduced by Haiman in his works [Hai99] and [Hai01] on Macdonald polynomials; it was proven in [Hai01] that B^n is normal, Cohen-Macauley and Gorenstein. It is an open problem if B^n has canonical or log-canonical singularities. In this work we partially answer these questions.

Apart from being interesting in its own, the investigation about the singularities of B^n is in tight relation with a number of interesting problems. The first and more immediate — which is one of the main motivations of this work — is the potential application to vanishing theorems, since sufficiently good singularities would allow the use of Kawamata-Viehweg or Kodaira vanishing over B^n; an example of this use already appeared in [Sca15, Section 5.2].

A second source of interest, which also offers an effective way to address the problem, is the link with the study of log-canonical thresholds of subspace arrangements. Since B^n is the blow-up of the big diagonal in X^n, it turns out that the scheme B^n — or, in other words, the pair $(B^n, \emptyset)$ — has exactly the same kind of singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$. Now, one can determine the kind of singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$ by studying its log-canonical threshold at each point. Since this problem is now local in nature, one can take X as the affine plane $\mathbb{C}^2$: in this case the big diagonal Δ_n can be thought as a subspace arrangement. This problem is similar with that of finding log-canonical thresholds of hyperplane arrangements, already studied and solved in [Mus06]. On the other hand, there are not many examples in literature of computations of log-canonical thresholds of arrangements of subspaces of higher codimension: an exception is the study of configurations of lines through the origin in $\mathbb{C}^3$ by Teitler [Tei07]. An important part of his work deals with the understanding of the embedded components that appear when pulling back the ideal of the configuration of lines to the blow-up of the origin in $\mathbb{C}^3$; the presence of embedded components is the main difficulty that hinders an explicit log-resolution of the ideal of the configuration.

The case of the pair $(X^n, \mathcal{I}_{\Delta_n})$ — for $X = \mathbb{C}^2$ — is similar because we deal with an arrangement of codimension 2 subspaces Δ_n in $\mathbb{C}^{2n}$, but it is more difficult because the complexity of the problem grows very rapidly with n. However, for $X = \mathbb{C}^2$, Haiman gave a precise description of a set of generators for the ideal $\mathcal{I}_{\Delta_n}$, from which we can deduce the order of the ideal $\mathcal{I}_{\Delta_n}$ at each point. As a consequence, we can establish the upper bound (proposition 2.9)

$$\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq \frac{2n-2}{d_n}$$

for the log-canonical threshold of the pair $(X^n, \mathcal{I}_{\Delta_n})$. Here d_n is the natural number defined in remark 2.7. We actually believe that the above inequality is in fact an equality (Conjecture 1). This would imply that the singularities of B^n are canonical if and only if $n \leq 7$, log-canonical if $n \leq 8$ and not log-canonical if $n \geq 9$ (Conjecture 2). We can actually prove — and this is the main result of this work —

Theorem 2.11. *The singularities of the isospectral Hilbert scheme are canonical if $n \leq 5$ and log-canonical if $n \leq 7$. For $n \geq 9$ they are not log-canonical.*

Not unexpectedly, this problem is in close relation with the geometry of the Hilbert scheme of points as well. Indeed, after a result by Song in [Son14], results about the pair $(X^n, \mathcal{I}_{\Delta_n})$ can be precisely translated into results about the pair $(X^{[n]}, \mathcal{I}_{\partial X^{[n]}})$, where $X^{[n]}$ is the Hilbert scheme of n points over X and $\partial X^{[n]}$ is its boundary. In particular the previous upper bound for $\mathrm{lct}(X^n, \mathcal{I}_{\Delta_n})$ impiies the upper bound $\mathrm{lct}(X^{[n]}, \mathcal{I}_{\partial X^{[n]}}) \leq (n-2)/d_n$. The mentioned conjecture on $\mathrm{lct}(X^n, \mathcal{I}_{\Delta_n})$ would imply that the last upper bound is actually an equality.

Finally, the problem of understanding the singularities of the isospectral Hilbert scheme should be a drive to the construction of an explicit $\mathfrak{S}_n$-equivariant log-resolution of B^n, or — what is equivalent — to an explicit $\mathfrak{S}_n$-equivariant log-resolution $f : Y \longrightarrow X^n$ of the pair $(X^n, \mathcal{I}_{\Delta_n})$. This would be a deep and importat result on many levels. Firstly, it would provide another important compactification of the configuration space $F(X, n) := X^n \setminus \Delta_n$ after the celebrated Fulton-MacPherson compactification $X[n]$ (see [FM94]): the latter is not, unfortunately, a log-resolution of the pair $(X^n, \mathcal{I}_{\Delta_n})$, since, when computing the pre-image of the ideal $\mathcal{I}_{\Delta_n}$ to $X[n]$ embedded components appear. Hence an explicit $\mathfrak{S}_n$-equivariant log-resolution of $(X^n, \mathcal{I}_{\Delta_n})$ might be built by further blowing-up the Fulton-MacPherson compactification in order to get rid of these components; however, it is a very difficult problem to track and control the embedded components that arise in this way.

Secondly, supposing that the stabilizers of the $\mathfrak{S}_n$-action on the resolution Y were trivial, then, passing to the quotient would provide an explicit resolution $\hat{f} : Y/\mathfrak{S}_n \longrightarrow S^n X$ of the symmetric variety. We mention that, in general, no such explicit resolution is known yet. In [Uly02] Ulyanov made a step forward proposing a refinement of the Fulton-MacPherson compactification in a way that the stabilizers of the natural $\mathfrak{S}_n$-action are abelian, and not just solvable.

Finally, such a resolution $f : Y \longrightarrow X^n$ might be useful in the understanding the geometry of ideal sheaves of subschemes supported in big diagonals of the form $\mathcal{O}(-\lambda\Delta)$, appeared in the work [Sca15].

In the final section of this article we provide two different log-resolutions of the pair $(X^3, \mathcal{I}_{\Delta_3})$, and hence of B^3: one crepant, the other $\mathfrak{S}_3$-equivariant.

We work over the field of complex numbers. By point we always mean a closed point.

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1 Singularities of pairs and log-canonical thresholds

Definition 1.1. [Kol97, Laz04] Let M be an irreducible complex algebraic variety, and $\mathfrak{a}$ an ideal sheaf of $\mathcal{O}_M$. A *log-resolution* of the pair $(M, \mathfrak{a})$ is a projective birational map $f : Y \longrightarrow M$ such that Y is nonsingular, the exceptional locus $\mathrm{exc}(f)$ is a divisor, the ideal sheaf $f^{-1}\mathfrak{a} := \mathfrak{a} \cdot \mathcal{O}_Y$ is equal to $\mathcal{O}_Y(-F)$, where F is an effective divisor on Y with the property that $F + \mathrm{exc}(f)$ has simple normal crossing support.

Definition 1.2. Let M be a complex algebraic variety, normal and irreducible; let K_M be its canonical divisor. Suppose that M is $\mathbb{Q}$-Gorenstein, that is, for some $r \in \mathbb{N}^*$, rK_M is Cartier. Let $\mathfrak{a}$ be an ideal sheaf of $\mathcal{O}_M$. Consider a log-resolution $f : Y \longrightarrow M$ of the pair $(M, \mathfrak{a})$. Then, as $\mathbb{Q}$-Cartier divisors,

$$K_Y - f^*(K_M) + f^{-1}(\mathfrak{a}) = \sum_i a_i E_i$$

where E_i are irreducible component of a simple normal crossing divisor and $a_i \in \mathbb{Q}$. We say that the singularities of the pair $(M, \mathfrak{a})$ are *canonical* if $a_i \geq 0$; *log-canonical* if $a_i \geq -1$.

Definition 1.3. Let M be a smooth algebraic variety and $\mathfrak{a}$ an ideal sheaf of $\mathcal{O}_M$. Let $c \in \mathbb{Q}$, $c > 0$. Let $f : Y \longrightarrow M$ be a log-resolution of the pair $(M, \mathfrak{a})$ and let F be the effective Cartier divisor on Y

such that $f^{-1}\mathfrak{a} = \mathcal{O}_Y(-F)$. Then the *multiplier ideal sheaf* $\mathcal{J}(c \cdot \mathfrak{a})$ associated to c and $\mathfrak{a}$ is the ideal sheaf of $\mathcal{O}_M$ defined as

$$\mathcal{J}(c \cdot \mathfrak{a}) := f_* \mathcal{O}_Y(K_{Y/M} - [c \cdot F]) \ ,$$

where $[c \cdot F]$ is the integral part of the $\mathbb{Q}$-divisor F. The definition just given does not depend on the choice of the log-resolution [Laz04]. For $x \in M$, the *log-canonical threshold* of the pair $(M, \mathfrak{a})$ at the point x is defined as

$$\operatorname{lct}_x(M, \mathfrak{a}) := \sup\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot \mathfrak{a})_x = \mathcal{O}_{M,x}\} = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot \mathfrak{a})_x \subset \mathfrak{m}_x\} \ .$$

Define, moreover, $\operatorname{lct}(M, \mathfrak{a}) := \inf_{x \in M} \operatorname{lct}_x(M, \mathfrak{a})$.

Remark 1.4. In the above definition of $\operatorname{lct}_x(M, \mathfrak{a})$ the inf are actually minima [Laz04, Example 9.3.16].

Proposition 1.5. *Let M be a smooth complex algebraic variety and let $\mathfrak{a}$ be an ideal sheaf of $\mathcal{O}_M$. Consider the blow-up $g : B := \operatorname{Bl}_{\mathfrak{a}} M \longrightarrow M$ of M along the ideal $\mathfrak{a}$, with exceptional divisor E. Suppose that B is irreducible, normal and Gorenstein; suppose moreover that $K_B = g^* K_M + E$. Then B has (log-) canonical singularities if and only if the pair $(M, \mathfrak{a})$ has.*

Proof. Let $h : Y \longrightarrow B$ be a log-resolution of the pair (B, E). Consider the map $f = g \circ h$. We claim that f is a log-resolution of the pair $(M, \mathfrak{a})$. Indeed $\operatorname{exc}(f)$ is divisorial, since f is a birational morphism between smooth varieties. Moreover, set-theoretically, $\operatorname{exc}(f) = \operatorname{exc}(h) \cup h^{-1}\operatorname{exc}(g) = \operatorname{exc}(h) \cup h^{-1}E$, which — since h is a log-resolution of (B, E) — is a divisor with snc support. Hence $\operatorname{exc}(f)$ is a divisor with snc support. Moreover $f^{-1}\mathfrak{a} = h^{-1}g^{-1}\mathfrak{a} = h^{-1}E$ is a Cartier divisor. Finally, as Cartier divisors, $\operatorname{exc}(f) + h^{-1}E$ coincides with $\operatorname{exc}(h) + 2h^{-1}E$, which has the same support as $\operatorname{exc}(f)$ and hence is a divisor with snc support. Then

$$K_Y - h^* K_B = K_Y - h^* g^* K_M - h^* E = K_Y - f^* K_M + f^{-1}\mathfrak{a}$$

which allows us to conclude. □

2 The isospectral Hilbert scheme

Definition 2.1. Let $n \in \mathbb{N}^*$. Let X be a smooth complex algebraic surface. Let Δ_n be the big diagonal in X^n, that is, Δ_n is the scheme-theoretic union of pairwise diagonals Δ_{ij}, $1 \leq i < j \leq n$. The *isospectral Hilbert scheme* B^n is the blow up of X^n along the big diagonal Δ_n.

Remark 2.2. It is well known that the isospectral Hilbert scheme B^n is irreducible, normal, Cohen-Macaulay and Gorenstein [Hai01].

2.1 The big diagonal in X^n

As an immediate consequence of proposition 1.5, we have a very precise correspondence between the singularities of the isospectral Hilbert scheme B^n and those of the pair $(X^n, \mathcal{I}_{\Delta_n})$.

Corollary 2.3. *The isospectral Hilbert scheme B^n has (log-) canonical singularities if and only if the pair $(X^n, \mathcal{I}_{\Delta_n})$ has (log-) canonical singularities.*

Remark 2.4. It is well known [Laz04, Example 9.3.16] that a pair $(M, \mathfrak{a})$ has log-canonical singularities if and only if $\operatorname{lct}(M, \mathfrak{a}) \geq 1$. On the other hand, if M is Gorenstein, then the discrepancies a_i are necessarily integers; consequently the pair $(M, \mathfrak{a})$ is canonical if and only if $\operatorname{lct}(M, \mathfrak{a}) > 1$, that is, if and only if $\mathcal{J}(M, \mathfrak{a}) = \mathcal{O}_M$. Hence we have that the isospectral Hilbert scheme B^n has canonical singularities if and only if $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) > 1$ or, equivalently, if $\mathcal{J}(X^n, \mathcal{I}_{\Delta_n})$ is trivial; the singularities of B^n are log-canonical if and only if $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1$.

Remark 2.5. The log-canonical threshold $\operatorname{lct}_x(M, \mathfrak{a})$ at the point $x \in M$ coincides with the *complex singularity exponent* $c_x(\mathfrak{a})$ of $\mathfrak{a}$ at the point x [DK01], which is an holomorphic invariant. As a consequence, the log-canonical threshold $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n})$ of the pair $(X^n, \mathcal{I}_{\Delta_n})$ for an arbitrary surface X is equal to the log-canonical threshold of the pair $((\mathbb{C}^2)^n, \mathcal{I}_{\Delta_n})$.

Remark 2.6 (Generators of $\mathcal{I}_{\Delta_n}$ for $X = \mathbb{C}^2$). In [Hai01] Haiman finds an explicit set of generators for ideal of the big diagonal Δ_n of $(\mathbb{C}^2)^n$. Write $(\mathbb{C}^2)^n$ as $\operatorname{Spec}\mathbb{C}[x_1, y_1, \dots, x_n, y_n]$. If $\bar{p}, \bar{q} \in \mathbb{N}^n$, denote with $\Delta(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ the $\mathfrak{S}_n$-anti-invariant regular function

$$\Delta(\bar{p}, \bar{q}, \bar{x}, \bar{y}) := \det(x_i^{p_i} y_j^{q_j})_{ij}$$

in the variables $x_1, \dots, x_n, y_1, \dots, y_n$. If there is no risk of confusion, we will drop the indication of the variables and we will just write it as $\Delta(\bar{p}, \bar{q})$. Haiman proves that homogeneous polynomials of the form $\Delta(\bar{p}, \bar{q})$ generate the ideal $\mathcal{I}_{\Delta_n}$. Of course the function $\Delta(\bar{p}, \bar{q})$ is non identically zero if and only if the points $(p_i, q_i) \in \mathbb{N} \times \mathbb{N}$ are all distinct.

Remark 2.7 (Generators of minimal degree in $\mathcal{I}_{\Delta_n}$). A nonzero homogeneous polynomial of the form $\Delta(\bar{p}, \bar{q})$ is of minimal degree if the set of points $\{(p_i, q_i), i = 1, \dots, n\}$ minimize the weight $\sum_i (p_i + q_i)$. Now for any $n \in \mathbb{N}$ there exist two natural numbers k and h, with $h < k$, uniquely determined by n, such that $n = k(k+1)/2 + h$. The integers k and h explain how to arrange n distinct points (p_i, q_i) in $\mathbb{N} \times \mathbb{N}$ in such a way that the weight $\sum_i (p_i + q_i)$ is the minimum possible: fill in the first antidiagonals in $\mathbb{N} \times \mathbb{N}$, of weight 0 to $k - 1$, with $k(k+1)/2$ points of nonnegative integral coordinates and on the antidiagonal of weight k put, in an arbitrary way, h points. Consequently, a generator of minimal degree has degree

$$d_n = \sum_{i=0}^{k-1} i(i+1) + hk = \frac{1}{3} k(k^2 + 3h - 1) \ .$$

Remark 2.8. Consider the diagonal Δ_n inside $(\mathbb{C}^2)^n = \operatorname{Spec}\mathbb{C}[x_1, y_1, \dots, x_n, y_n]$ and consider its ideal $\mathcal{I}_{\Delta_n} \subseteq \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$. We build now a new coordinate system, in the following way. Consider the vector space $(\mathbb{C}^2)^{n-1}$ with coordinates $(z_1, w_1, \dots, z_{n-1}, w_{n-1})$ and $\mathbb{C}^2$ with coordinates (α, β). Consider now the isomorphism

$$\varphi : (\mathbb{C}^2)^n \longrightarrow (\mathbb{C}^2)^{n-1} \times \mathbb{C}^2 \tag{2.1}$$

defined by the coordinate change

$$z_i = x_1 - x_{i+1} \ , \qquad w_i = y_1 - y_{i+1} \qquad \text{for } i = 1, \dots n-1$$
$$\alpha = \sum_{i=1}^{n} x_i \ , \qquad \beta = \sum_{i=1}^{n} y_i$$

In the new coordinates the pairwise diagonals in $(\mathbb{C}^2)^n$ are now given by ideals (z_i, w_i) and $(z_i - z_j, w_i - w_j)$, $1 \le i < j \le n-1$ and the ideal $\mathcal{I}_{\Delta_n}$ is the intersection

$$\mathcal{I}_{\Delta_n} = \cap_{i=1}^{n-1} (z_i, w_i) \bigcap \cap_{1 \le i < j \le n-1} (z_i - z_j, w_i - w_j)$$

inside $\mathbb{C}[z_1, w_1, \dots, z_{n-1}, w_{n-1}, \alpha, \beta]$. Since the generators of $\mathcal{I}_{\Delta_n}$ are just polynomials in the z_i, w_i, the ideal $\mathcal{I}_{\Delta_n}$ is the extension of an ideal $\mathcal{I}_{\widetilde{D}_{n-1}} \subseteq \mathbb{C}[z_1, \dots, z_{n-1}, w_1, \dots, w_{n-1}]$, generated by the same elements. In other words, we can write

$$\mathcal{I}_{\Delta_n} \simeq \varphi^*(\mathcal{I}_{\widetilde{D}_{n-1}} \boxtimes \mathcal{O}_{\mathbb{C}^2}) \ . \tag{2.2}$$

Consider now the projection $r : (\mathbb{C}^2)^{n-1} \times \mathbb{C}^2 \longrightarrow (\mathbb{C}^2)^{n-1}$. Under the identification φ, the small diagonal $\Delta_{1,\dots,n}$ in $(\mathbb{C}^2)^n$ is the pre-image $r^{-1}(\{0\})$ by r of the origin $\{0\}$ in $(\mathbb{C}^2)^{n-1}$. Consequently, the order of the big diagonal Δ_n along the small diagonal $\Delta_{1,\dots,n}$ coincide with the order of $\widetilde{D}_{n-1}$ at the origin: $\operatorname{ord}_{\Delta_{1,\dots,n}} \mathcal{I}_{\Delta_n} = \operatorname{ord}_0 \mathcal{I}_{\widetilde{D}_{n-1}}$; but $\operatorname{ord}_0 \mathcal{I}_{\widetilde{D}_{n-1}}$ is the minimal degree of generators of $\mathcal{I}_{\widetilde{D}_{n-1}}$. But $\mathcal{I}_{\Delta_n}$ and $\mathcal{I}_{\widetilde{D}_{n-1}}$ have the same generators, hence $\operatorname{ord}_{\Delta_{1,\dots,n}} \mathcal{I}_{\Delta_n} = d_n$. Since the order of a coherent ideal along a subvariety is an holomorphic invariant, we can say in general that, for a smooth algebraic surface X,

$$\operatorname{ord}_{\Delta_{1,\dots,n}} \mathcal{I}_{\Delta_n} = d_n \ .$$

2.2 F-pure thresholds

For computational convenience we consider the characteristic p analogue of the log-canonical threshold. Let k be a perfect field of characteristic p and let R be a finitely generated regular k-algebra and let $\mathfrak{a} \subseteq R$ a nonzero ideal; consider $M = \operatorname{Spec} R$ and $x \in V(\mathfrak{a})$ a closed point corresponding to a maximal ideal $\mathfrak{m}_x$. For $e \in \mathbb{N}^*$, define

$$\nu_{\mathfrak{a}}(e) := \max\left\{ i \in \mathbb{N} \mid \mathfrak{a}^i \not\subseteq \mathfrak{m}_x^{[p^e]} \right\}$$

where $\mathfrak{m}_x^{[p^e]}$ is the ideal generated by p^e-powers of generators of $\mathfrak{m}_x$. The inequality $\nu_{\mathfrak{a}}(e+1) \geq p\nu_{\mathfrak{a}}(e)$ implies that the sequences $\nu_{\mathfrak{a}}(e)/p^e$ and $\nu_{\mathfrak{a}}(e)/(p^e-1)$ are nondecreasing [MTW05, Lemma 1.1]. The F-pure threshold of the ideal $\mathfrak{a}$ at the point x is defined as

$$\operatorname{fpt}_x(M, \mathfrak{a}) := \lim_{e \to +\infty} \frac{\nu_{\mathfrak{a}}(e)}{p^e} = \lim_{e \to +\infty} \frac{\nu_{\mathfrak{a}}(e)}{(p^e-1)} = \sup_{e \in \mathbb{N}^*} \frac{\nu_{\mathfrak{a}}(e)}{p^e} = \sup_{e \in \mathbb{N}^*} \frac{\nu_{\mathfrak{a}}(e)}{(p^e-1)} \ . \tag{2.3}$$

Suppose now that $\mathfrak{a}$ is principal: we write simply $\nu_f(e)$ instead of $\nu_{(f)}(e)$ and $\operatorname{fpt}_x(M, f)$ instead of $\operatorname{fpt}_x(M, (f))$. In this case the sequence $\nu_{\mathfrak{a}}(e)/p^e$ is bounded above by 1. Hence, for any $e \in \mathbb{N}^*$ we have the inequalities

$$\frac{\nu_f(e)}{(p^e-1)} \leq \operatorname{fpt}_x(M, f) \leq 1 \ . \tag{2.4}$$

Suppose now that M is the affine space $\mathbb{A}^n_{\mathbb{Z}}$ over $\mathbb{Z}$ and $\mathfrak{a}$ is a nonzero ideal of $R := \mathbb{Z}[x_1, \ldots, x_n]$. For any prime p consider the mod p reduction $M_p := \operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{F}_p)$ and $\mathfrak{a}_p = \mathfrak{a} \cdot \mathbb{F}_p[x_1, \ldots, x_n]$. On the other hand, if $\mathbb{K}$ is an arbitrary field extension of $\mathbb{Q}$ we can consider the extensions $\mathfrak{a}_{\mathbb{K}}$ inside $\mathbb{K}[x_1, \ldots, x_n]$, respectively and $M_{\mathbb{K}} := \operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{K})$. For varieties defined over arbitrary perfect fields, Zhu recently proved an interpretation of the log-canonical threshold in terms of dimensions of jet-schemes [Zhu13, Theorem B]; this result yields, as a consequence, the inequality $\operatorname{fpt}_x(M_p, \mathfrak{a}_p) \leq \operatorname{lct}_x(M_{\mathbb{Q}}, \mathfrak{a}_{\mathbb{Q}})$ for every prime p and for every closed point $x \in V(\mathfrak{a})$ [Zhu13, Corollary 4.2]. Since the dimension of a scheme does not change upon extension of the field of definition [Gro65, Corollaire 4.1.4], we have, for every prime p and any closed point $x \in V(\mathfrak{a})$

$$\operatorname{fpt}_x(M_p, \mathfrak{a}_p) \leq \operatorname{lct}_x(M_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}) \ . \tag{2.5}$$

2.3 Singularities of the isospectral Hilbert scheme

We begin with the following upper bound for the log-canonical threshold of the pair $(X^n, \mathcal{I}_{\Delta_n})$.

Proposition 2.9. *The log-canonical threshold of $(X^n, \mathcal{I}_{\Delta_n})$ is bounded above by $(2n-2)/d_n$:*

$$\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq \frac{2n-2}{d_n} \ .$$

Proof. By remark 2.5 it is sufficient to prove the inequality when $X = \mathbb{C}^2$. By remark 2.8, for $c \in \mathbb{Q}$, $c > 0$, the order of $c \cdot \mathcal{I}_{\Delta_n}$ along the small diagonal $\Delta_{1,\ldots,n}$ is cd_n; as soon as $cd_n \geq \operatorname{codim}_X \Delta_{1,\ldots,n} + 1 - 1 = 2n-2$, that is, if $c \geq (2n-2)/d_n$, by [Laz04, Example 9.3.7] we have that $\mathcal{J}(X, c \cdot \mathcal{I}_{\Delta_n}) \subseteq \mathcal{I}_{\Delta_{1,\ldots,n}}$. By definition of log-canonical threshold $\operatorname{lct}_0(X^n, \mathcal{I}_{\Delta_n})$ as infimum, we get the desired inequality $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq \operatorname{lct}_0(X^n, \mathcal{I}_{\Delta_n}) \leq (2n-2)/d_n$. □

Remark 2.10. Consider the symmetric variety $S^n X$, where X is a smooth complex algebraic surface; we will indicate with $\pi : X^n \longrightarrow S^n X$ the quotient projection. It is well known that $S^n X$ admits a stratification in strata $S^n_\lambda X$, where λ is a partition of n. The stratum $S^n_\lambda X$ is the locally closed subset of 0-cycles of the form $\sum_{i=1}^{l(\lambda)} \lambda_i x_i$, where $l(\lambda)$ is the length of the partition λ and x_i are $l(\lambda)$ distinct points in X. By means of this stratification of $S^n X$ we can define a stratification of X^n setting the stratum X^n_λ as the locally closed subset $\pi^{-1}(S^n_\lambda X)$. It is clear that if $x \in X^n_\lambda$ then a sufficiently small open set V_1 of x in X^n in the standard topology is biholomorphic to a sufficiently small open set V_2 of the origin in $(\mathbb{C}^2)^n$ of the form $V_2 = U_1^{\lambda_1} \times \cdots \times U_{l(\lambda)}^{\lambda_{l(\lambda)}}$, where U_i are adequate small open sets of the origin in $\mathbb{C}^2$, such that, via the biholomorphic map, the ideal $\mathcal{I}_{\Delta_n}$ over V_1 is sent to $\mathcal{I}_{\Delta_{\lambda_1}} \boxtimes \cdots \boxtimes \mathcal{I}_{\Delta_{\lambda_{l(\lambda)}}}$ over V_2. Therefore, if $x \in X^n_\lambda$, we have, by proposition 2.9 and by [Laz04, Proposition 9.5.22] that

$$\operatorname{lct}_x(X^n, \mathcal{I}_{\Delta_n}) = \min\left\{ \operatorname{lct}_0((\mathbb{C}^2)^{\lambda_i}, \mathcal{I}_{\Delta_{\lambda_i}}) \mid i = 1, \ldots, l(\lambda) \right\} \leq \frac{2\lambda_1 - 2}{d_{\lambda_1}} \ . \tag{2.6}$$

We now make the following conjecture

Conjecture 1. *If a point x of X^n lies in the stratum X^n_λ, where λ is a partition of n, then* $\operatorname{lct}_x(X^n, \mathcal{I}_{\Delta_n}) = (2\lambda_1 - 2)/d_{\lambda_1}$*. Therefore*

$$\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) = \frac{2n-2}{d_n} .$$

This conjecture would immediately imply the following fact about the singularities of the isospectral Hilbert scheme B^n.

Conjecture 2. *The singularities of the isospectral Hilbert scheme B^n are canonical if and only if $n \leq 7$, log-canonical if $n \leq 8$, not log-canonical if $n \geq 9$.*

We are able to partially prove conjecture 2.

Theorem 2.11. *The singularities of the isospectral Hilbert scheme are canonical if $n \leq 5$, log-canonical if $n \leq 7$. For $n \geq 9$ they are not log-canonical.*

Proof. By corollary 2.3 and by remark 2.4 the singularities of the isospectral Hilbert scheme are log-canonical if and only if $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1$ and canonical if and only if $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \gneq 1$. For $n \geq 9$, by proposition 2.9, $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \leq (2n-2)/d_n \leq 16/17$. Hence they can't be log-canonical.

Let's now prove the first statement. Using corollary 2.3 and remark 2.4 it is sufficient to prove that the singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$ are canonical for $n \leq 5$ and that $\operatorname{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1$ for $n = 6, 7$. By remark 2.5 it is sufficient to prove these facts for $X = \mathbb{C}^2$. By (2.2), it is then sufficient to prove that the pair $(\mathbb{C}^{2n-2}, \mathcal{I}_{\widetilde{D}_{n-1}})$ has canonical singularities for $n \leq 5$ and is log-canonical for $n = 6, 7$.

To prove that the pair $(\mathbb{C}^{2n-2}, \mathcal{I}_{\widetilde{D}_{n-1}})$ is canonical for $n \leq 4$ we will use Kollar-Bertini theorem [Kol97, Theorems 4.5, 4.5.1], [Laz04, Example 9.3.50]: in other words we will find a $g \in \mathcal{I}_{\widetilde{D}_{n-1}}$ such that $\operatorname{div} g$ has rational (or canonical) singularities; then Kollar-Bertini theorem implies that the pair $(\mathbb{C}^{2n-2}, \mathcal{I}_{\widetilde{D}_{n-1}})$ is canonical. For $n = 3$ such a g can be chosen as the generator of minimal degree of $\mathcal{I}_{\widetilde{D}_2}$, that is, $g = z_1 w_2 - z_2 w_1$: it defines an affine quadric cone of in $\mathbb{C}^4$ projecting a smooth quadric in $\mathbb{P}^3$ from the origin of $\mathbb{C}^4$. Hence, by [Bur74, Example 1.2], it has rational singularities. For $n = 4$ we can use the generator of minimal degree of $\mathcal{I}_{\widetilde{D}_3}$ given by the polynomial $g = \Delta((1,0,1),(0,1,1),\bar{z},\bar{w})$. One can show that g has rational singularities using *Macaulay2* [GS] and, in particular, the command `hasRationalSing` of the package `D-modules`.

For $n \geq 5$ it is computationally more efficient to use characteristic p methods. Let now $n = 5$. By the equality in (2.6) and by what we just proved, we know that for any point x in a strata X^5_λ, with $\lambda \neq (5)$, we have $\operatorname{lct}_x(X^5, \mathcal{I}_{\Delta_5}) \geq \operatorname{lct}(X^4, \mathcal{I}_{\Delta_4}) > 1$. It is then sufficient to prove that, for a point $x \in \Delta_{1,\dots,5}$, $\operatorname{lct}_x(\mathbb{C}^{10}, \mathcal{I}_{\Delta_5}) > 1$. Because of the isomorphism (2.2) it is sufficient to prove that $\operatorname{lct}_0(\mathbb{C}^8, \mathcal{I}_{\widetilde{D}_4}) > 1$. By (2.5) it is sufficient to prove, for some prime p, that $\operatorname{fpt}_0((\mathbb{F}_p^2)^4, (\mathcal{I}_{\widetilde{D}_4})_p) > 1$. Consider the polynomials $g = \Delta((1,0,2,1),(0,1,0,2),\bar{z},\bar{w})$ and $h = \Delta((1,0,2,0),(0,1,0,2),\bar{z},\bar{w})$ in $\mathcal{I}_{\widetilde{D}_4}$; we can check, using *Macaulay2* and passing modulo $p = 7$, that the class of $g^2 h^5$ is nonzero in $\mathbb{F}_7[z_1,\dots,z_4,w_1,\dots,w_4]/\mathfrak{m}_0^{[7]}$, thus proving that $\nu_{\mathfrak{a}}(1) \geq 7$, where $\mathfrak{a} = (\mathcal{I}_{\widetilde{D}_4})_7$, and hence that $\operatorname{fpt}_0((\mathbb{F}_7^2)^4, (\mathcal{I}_{\widetilde{D}_4})_7) \geq 7/6 > 1$, by (2.3). Therefore the pair $(X^5, \mathcal{I}_{\Delta_5})$ has canonical singularities.

Let now $n = 6, 7$. By the equality in (2.6) and by what we just proved, we already know that for any point x in a stratum X^n_λ, with $\lambda \neq (6)$ — in the case $n = 6$ — or $\lambda \neq (7)$ and $\lambda \neq (6,1)$ — in the case $n = 7$ — we have $\operatorname{lct}_x(X^n, \mathcal{I}_{\Delta_n}) \geq \operatorname{lct}(X^5, \mathcal{I}_{\Delta_5}) > 1$. For $n = 6$ it is then sufficient to prove that $\operatorname{lct}_x(\mathbb{C}^{12}, \mathcal{I}_{\Delta_6}) > 1$ when $x \in \Delta_{1,\dots,6}$; by the isomorphism (2.2), it is sufficient to prove that $\operatorname{lct}_0(\mathbb{C}^{10}, \mathcal{I}_{\widetilde{D}_5}) \geq 1$; once we prove it, it is sufficient to prove that $\operatorname{lct}_x(\mathbb{C}^{14}, \mathcal{I}_{\Delta_7}) > 1$ for $x \in \Delta_{1,\dots,7}$, or equivalenty, after (2.2), that $\operatorname{lct}_0(\mathbb{C}^{12}, \mathcal{I}_{\widetilde{D}_6}) \geq 1$. By (2.5) it is sufficient to prove, for some prime p, that $\operatorname{fpt}_0((\mathbb{F}_p^2)^{n-1}, (\mathcal{I}_{\widetilde{D}_{n-1}})_p) \geq 1$ for $n = 6, 7$. By the first of the inequalities (2.4) it is then sufficient to find a polynomial $g \in \mathcal{I}_{\widetilde{D}_{n-1}}$, with integral coefficients, such that, for some prime p, $\nu_{g_p}(1) = p - 1$ at the origin: here, for a polynomial g with integral coefficients, we denote with g_p its mod p reduction in in $(\mathcal{I}_{\widetilde{D}_{n-1}})_p$. Consider the polynomials with integral coefficients $g = \Delta((1,0,2,1,0),(0,1,0,1,2),\bar{z},\bar{w})$, for $n = 6$, and $h = \Delta((1,0,2,1,0,2),(0,1,0,1,2,1),\bar{z},\bar{w})$, for $n = 7$. Then, using *Macaulay2* and passing modulo 7, we checked that the class of g_7^6 in $\mathbb{F}_7[z_1,\dots,z_5,w_1,\dots,w_5]/\mathfrak{m}_0^{[7]}$ and h_7^6 in $\mathbb{F}_7[z_1,\dots,z_6,w_1,\dots,w_6]/\mathfrak{m}_0^{[7]}$ are both non zero. This proves that, choosing the prime 7, $\nu_{g_7}(1) = 6 = \nu_{h_7}(1)$ and hence $\operatorname{fpt}_0((\mathbb{F}_7^2)^5, g_7) = 1$, in case $n = 6$, and $\operatorname{fpt}_0((\mathbb{F}_7^2)^6, h_7) = 1$, in case $n = 7$, and we can conclude. $\square$

2.4 Relation with the geometry of the Hilbert scheme of points

The geometry of the pair $(X^n, \mathcal{I}_{\Delta_n})$ is not only directly related to the geometry of the isospectral Hilbert scheme B^n, but also to the geometry of the Hilbert scheme of n points $X^{[n]}$ over the surface X. Consider the boundary $\partial X^{[n]}$ of $X^{[n]}$. Song proved in [Son14, Proposition 4.3.5] that

$$\mathrm{lct}(X^{[n]}, \mathcal{I}_{\partial X^{[n]}}) = \mathrm{lct}(S^n X, \mathcal{I}_{\Delta_n}^{\mathfrak{S}_n}) = \frac{1}{2}\,\mathrm{lct}(X^n, \mathcal{I}_{\Delta_n}) \ .$$

Hence proposition 2.9 implies immediately the

Corollary 2.12. *The log-canonical threshold of the pair $(X^{[n]}, \mathcal{I}_{\partial X^{[n]}})$ is bounded above by $(n-1)/d_n$.*

Moreover, conjecture 1 would imply

Conjecture 3. *The log-canonical threshold of the pair $(X^{[n]}, \mathcal{I}_{\partial X^{[n]}})$ is precisely given by $(n-1)/d_n$.*

3 Two resolutions of B^3

The aim of this subsection is two provide two explicit resolutions of singularities of B^3; the first will be *crepant*, the second will be $\mathfrak{S}_3$*-equivariant*. We begin with some remarks and technical lemmas.

Remark 3.1. Let M a smooth algebraic variety and let F be a coherent sheaf over M. We recall that an integral subscheme V of M is called a *prime cycle associated to* F if there exists an invertible coherent $\mathcal{O}_V$-module L and an embedding $L \hookrightarrow F$ of coherent $\mathcal{O}_M$-modules.

Remark 3.2. Let M be a smooth algebraic variety and Y a smooth subvariety. Let $Z \subseteq M$ be a closed subscheme, defined by the ideal sheaf $\mathcal{I}_Z$. Let $r = \mathrm{ord}_Y \mathcal{I}_Z$ the order of Z along Y. Consider the blow-up $f : \mathrm{Bl}_Y M \longrightarrow M$ of Y in M and denote with E its exceptional divisor. The *weak transform* $\widetilde{Z}$ of Z in $\mathrm{Bl}_Y M$ is defined by the residual ideal $\mathcal{I}_{\widetilde{Z}} := (\mathcal{I}_{f^{-1}(Z)} : \mathcal{I}_E^r)$. The ideal of the total transform $f^{-1}(Z)$ is then given by the product

$$\mathcal{I}_{f^{-1}(Z)} = \mathcal{I}_E^r \cdot \mathcal{I}_{\widetilde{Z}} \ .$$

It is well known that the weak transform does not necessarily coincide with the strict transform $\widehat{Z}$; in general one just has that $\mathcal{I}_{\widetilde{Z}} \subseteq \mathcal{I}_{\widehat{Z}}$, and that the two ideals coincide outside the exceptional divisor. Indeed the weak transform $\widetilde{Z}$ could contain embedded components over the exceptional divisor, while the strict transform doesn't. This is, in any case, the only possible difference between $\widetilde{Z}$ and $\widehat{Z}$, as the next criterion proves.

Proposition 3.3. *Let M be a smooth algebraic variety and Y a smooth subvariety. Let $Z \subseteq M$ be a closed subscheme. Consider the blow-up map $f : \mathrm{Bl}_Y M \longrightarrow M$ and let E be the exceptional divisor. Then the weak transform $\widetilde{Z}$ of Z coincide with the strict transform $\widehat{Z}$ if and only if E does not contain any prime cycle associated to $\widetilde{Z}$. In this case, for any positive integer l, the subschemes lE and $\widehat{Z}$ are transverse.*

Proof. The necessity of the condition is clear. We just have to prove the sufficiency. Recall that the strict transform $\widehat{Z}$ can be identified with the blow-up $\mathrm{Bl}_{Y\cap Z} Z$: this is a consequence, for example, of [EH00, Proposition IV-21]. Indicate with λ the canonical section of $\mathcal{O}_{\mathrm{Bl}_Y M}(E)$. We have that E does not contain prime cycles associated to $\widetilde{Z}$ if and only if the morphism $\lambda : \mathcal{O}_{\widetilde{Z}}(-E) \longrightarrow \mathcal{O}_{\widetilde{Z}}$ is injective. In this case the ideal $\mathcal{I}_{\widetilde{Z}\cap E/\widetilde{Z}}$ of $\widetilde{Z} \cap E$ in $\widetilde{Z}$ is an invertible ideal of $\mathcal{O}_{\widetilde{Z}}$. Hence the map $f\big|_{\widetilde{Z}} : \widetilde{Z} \longrightarrow Z$ factors via the blow-up $\mathrm{Bl}_{Y\cap Z} Z$, that is, via the strict transform $\widehat{Z}$. Hence we have the injection of schemes $\widetilde{Z} \hookrightarrow \widehat{Z}$. But it is always true that $\widehat{Z} \subseteq \widetilde{Z}$. Hence the weak transform coincides with the strict one. In this case, for any fixed positive integer l, the morphism $\lambda^l : \mathcal{O}_{\widehat{Z}}(-lE) \longrightarrow \mathcal{O}_{\widehat{Z}}$ is injective. Since $R^\bullet := 0 \longrightarrow \mathcal{O}_{\mathrm{Bl}_Y M}(-lE) \longrightarrow \mathcal{O}_{\mathrm{Bl}_Y M}$ is a locally free resolution of $\mathcal{O}_{lE}$, we can compute $\mathrm{Tor}_j(\mathcal{O}_{lE}, \mathcal{O}_{\widehat{Z}})$ as of the $(-j)$-cohomology of the complex $R^\bullet \otimes \mathcal{O}_{\widehat{Z}}$, which is $0 \longrightarrow \mathcal{O}_{\widehat{Z}}(-lE) \xrightarrow{\lambda^l} \mathcal{O}_{\widehat{Z}} \longrightarrow 0$. Hence $\mathrm{Tor}_j(\mathcal{O}_{lE}, \mathcal{O}_{\widehat{Z}}) = 0$ for $j > 0$. □

Remark 3.4. Let M be a smooth algebraic variety, and Y a smooth subvariety. Consider the blow-up map $f : \mathrm{Bl}_Y M \longrightarrow M$. Let H be an hypersurface in M. Then its weak and strict transform in $\mathrm{Bl}_Y M$ coincide.

Proof. Let E be the exceptional divisor. The weak transform $\widetilde{H}$ is a divisor whose associated prime cycles are the irreducible components of $\widetilde{H}$. Since, by definition of $\widetilde{H}$, one has that $E \not\subset \widetilde{H}$, then $\operatorname{codim}_{\mathrm{Bl_Y M}} E \cap \widetilde{H} = 2$ and hence the local equations of E and $\widetilde{H}$ define a regular sequence; hence E does not contain any prime cycles relative to $\widetilde{H}$. Hence $\widetilde{H} = \widehat{H}$. □

Lemma 3.5. *Let M be a smooth algebraic variety and let Y, W, Z three subschemes of M, such that Y is closed, W is integral and that $Y \not\subseteq W$. Let $\widehat{W}$, $\widehat{Z}$ be the strict transforms of W and Z inside $\mathrm{Bl}_Y M$. Then $\operatorname{ord}_W \mathcal{I}_Z = \operatorname{ord}_{\widehat{W}} \mathcal{I}_{\widehat{Z}}$.*

Proof. Note that if S, T are two subschemes of a smooth algebraic variety V, with T integral, then $\operatorname{ord}_T \mathcal{I}_S$ can be characterized as $\operatorname{ord}_T \mathcal{I}_S = \max\{n \in \mathbb{N} \mid \mathcal{I}_{S,T} \subseteq \mathfrak{m}_T^n\}$ where $\mathfrak{m}_T$ is the maximal ideal of the local ring $\mathcal{O}_{V,T}$ — that is, the ring of regular functions g defined on some open set U intersecting T [Har77, Exercise 3.13]— and where $\mathcal{I}_{S,T}$ is the ideal of functions g in $\mathcal{O}_{V,T}$ vanishing over $S \cap U$, if U is the open set of definition of g. Now the blow-up map $f : \mathrm{Bl}_Y M \longrightarrow M$ induces an isomorphism of local rings $f_W^* : \mathcal{O}_{M,W} \longrightarrow \mathcal{O}_{\mathrm{Bl}_Y M, \widehat{W}}$ under which $\mathcal{I}_{Z,W}$ is sent onto $\mathcal{I}_{\widehat{Z},\widehat{W}}$, hence the statement. □

Lemma 3.6. *Let M be a smooth algebraic variety of dimension at least 3; let H be a smooth hypersurface in M and W_1, W_2 two smooth subvarieties of M contained in H and transverse inside H. Consider now the composition f of blow-ups*

$$f : B := \mathrm{Bl}_{\widehat{W_2}} \mathrm{Bl}_{W_1} M \xrightarrow{f_2} \mathrm{Bl}_{W_1} M \xrightarrow{f_1} M \ ,$$

where $\widehat{W_2}$ is the strict transform of W_2 inside $\mathrm{Bl}_{W_1} M$. Denote with E_{W_1} the exceptional divisor of $\mathrm{Bl}_{W_1} M$ and with $E_{\widehat{W_2}}$ that of $\mathrm{Bl}_{\widehat{W_2}} \mathrm{Bl}_{W_1} M$. Then f is an isomorphism outside $f^{-1}(W_1 \cup W_2)$; moreover

$$f^{-1}(\mathcal{I}_{W_1 \cup W_2}) = \mathcal{I}_{\widehat{E_{W_1}}} \cdot \mathcal{I}_{E_{\widehat{W_2}}} = \mathcal{O}_B(-\widehat{E_{W_1}} - E_{\widehat{W_2}}) \ .$$

Finally the relative canonical bundle $K_{B/M}$ is isomorphic to $\mathcal{O}_B(\widehat{E_{W_1}} + E_{\widehat{W_2}})$.

Proof. In the particular case in which $M = \mathbb{C}^3$; $\mathcal{I}_H = (x)$; $\mathcal{I}_{W_1} = (x, y)$; $\mathcal{I}_{W_2} = (x, z)$ and hence $\mathcal{I}_{W_1 \cup W_2} = (x, yz)$, the statement can be proved by an explicit computation in coordinates, which we leave to the reader.

Le't now pass to the general case. Consider a point p in the intersection $W_1 \cap W_2$. Over an adequate open neighbourhood U of p in the standard complex topology, we can find local holomorphic coordinates x, y, z such that H is defined (over U) by the zeros of x, and W_1 and W_2 by the ideals (x, y) and (x, z), respectively. Alternatively, one can find an adequate affine neighbourhood U of p and regular function x, y, z over U such that the differentials dx, dy, dz are independent in $\mathfrak{m}_q/\mathfrak{m}_q^2$ for all $q \in U$ and such that H, W_1, W_2 are defined by ideals of the regular functions (x), (x, y) and (x, z) as in the holomorphic case. Hence the general situation can be obtained locally from the particular one above by a smooth base change: the statement follows. □

Lemma 3.7. *Let M be a smooth algebraic variety, H a smooth hypersurface of M, and W and Q two codimension 2 smooth subvarieties of M such that $Q \subseteq H$, $W \cap H \subseteq Q$ and $W \cap H$ is a smooth codimension 3 subvariety of M. Consider the blow-up $f : \mathrm{Bl}_W M \longrightarrow M$ of W in M, with exceptional divisor E_W. Then*

$$f^{-1}(\mathcal{I}_W \cap \mathcal{I}_Q) = \mathcal{I}_{E_W} \cdot \mathcal{I}_{\widehat{Q}} = \mathcal{I}_{E_W} \cap \mathcal{I}_{\widehat{Q}}$$

where $\widehat{Q}$ denote the strict transform of Q in $\mathrm{Bl}_W M$.

Proof. The statement is local in nature, over the base M: hence, by placing ourserlves on a small open neighbourhood of a point $p \in W \cap H$ in the complex topology, equipped with some holomorphic coordinates $(x, y, z, w_1, \ldots, w_r)$, we can suppose that the ideals of H, W and Q are given locally by $\mathcal{I}_H = (z)$, $\mathcal{I}_W = (x, y)$, $\mathcal{I}_Q = (x, z)$. Then $\mathcal{I}_W \cap \mathcal{I}_Q = (x, yz)$; the proof of the statement is now achieved through an easy computation in coordinates. □

3.1 A crepant resolution of B^3.

Conjecture 1 states that the log-canonical threshold of the pair $(X^3, \mathcal{I}_{\Delta_3})$ is 2. This fact suggests that B^3 might admit a crepant resolution. This is indeed the case, as we will prove in this subsection.

Remark 3.8. Let X be a smooth algebraic surface. If Y is any smooth variety admitting a projective birational morphism $f : Y \longrightarrow X^n$ over X^n such that such that $f^{-1}(\mathcal{I}_{\Delta_n})$ is an invertible ideal sheaf of $\mathcal{O}_Y$, then, by the universal property of the blow-up, the map f factors via the isospectral Hilbert scheme B^n as

$$\begin{array}{ccc} Y & & \\ {\scriptstyle h}\downarrow & \overset{f}{\searrow} & \\ B^n & \xrightarrow{\ p\ } & X^n \end{array}$$

providing a resolution h of B^n such that

$$K_Y - h^* K_{B^n} = K_Y - h^*(p^* K_{X^n} + E) = K_Y - f^* K_{X^n} - h^* E = K_Y - f^* K_{X^n} + f^{-1}(\mathcal{I}_{\Delta_n}) \ .$$

Remark 3.9. By the previous remark, in order to find a crepant resolution of B^n, it is sufficient to build a smooth variety Y and a projective birational map $f : Y \longrightarrow X^n$ such that $f^{-1}(\mathcal{I}_{\Delta_n})$ is an invertible ideal isomorphic to the relative anticanonical $-K_{Y/X^n} = f^* K_{X^n} - K_Y$.

Remark 3.10. The questions posed in the previous two remarks are local over the base and analytical in nature. Hence, to find a resolution of B^n in general, it is sufficient to find a smooth variety Y and a birational map as in the remark 3.8 for $X = \mathbb{C}^2$. Moreover, since in the identification (2.2), the ideal sheaf $\mathcal{I}_{\Delta_n}$ corresponds to $\mathcal{I}_{\widetilde{D}_{n-1}} \boxtimes \mathcal{O}_{\mathbb{C}^2}$, by flat base change it is sufficient to find a smooth variety Y and a projective birational morphism $f : Y \longrightarrow (\mathbb{C}^2)^{n-1}$ such that $f^{-1}(\mathcal{I}_{\widetilde{D}_{n-1}})$ is an invertible ideal. The resolution thus built will be crepant if and only if $f^{-1}(\mathcal{I}_{\widetilde{D}_{n-1}})$ is isomorphic to the anticanonical $-K_Y$.

For brevity's sake, in what follows, we will indicate the affine space $(\mathbb{C}^2)^2$ with V, the subscheme $\widetilde{D}_2$ with W. Fix coordinates (x, y, z, w) over V. The irreducible components of the subscheme W are linear subspaces W_1, W_2, W_3, defined by the ideals $I_1 = (x, y)$, $I_2 = (z, w)$, $I_3 = (x - z, y - w)$. The ideal $\mathcal{I}_W$ is then given by $\langle q, I_1 I_2 I_3 \rangle$, where q is the quadric $q = xw - yz$.

Proposition 3.11. *The projective birational morphism $f : Y \longrightarrow V$, defined as the composition of smooth blow-ups*

$$Y = Y_3 \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} V$$

where $Y_1 = \mathrm{Bl}_{W_1} V$, $Y_2 = \mathrm{Bl}_{\widehat{W}_2} Y_1$, $Y_3 = \mathrm{Bl}_{\widehat{\widehat{W}}_3} Y_2$, where $\widehat{W}_2, \widehat{\widehat{W}}_3$ are the strict transforms of W_2, W_3 in Y_1, Y_2, respectively, is an isomorphism outside the locus $f^{-1}(W)$. Moreover, the ideal sheaf $f^{-1}(\mathcal{I}_W)$ is invertible and isomorphic to $-K_Y$.

Proof. As generators of the ideal $\mathcal{I}_W$ we can choose the polynomials $q, xz(x - z), xw(y - w), yw(x - z), yw(y - w)$. Consider the first blow-up $Y_1 = \mathrm{Bl}_{W_1} V \simeq \mathrm{Bl}_0(\mathbb{C}^2) \times \mathbb{C}^2$ and denote with E_1 the exceptional divisor. We can write globally

$$x = \lambda u \ , \qquad y = \lambda v$$

where λ is the canonical section of $\mathcal{O}_{Y_1}(E_1)$ and u, v are homogeneous coordinates, thought as a basis in $H^0(\mathcal{O}_{Y_1}(-E_1))$. By definition of weak transform we have $\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_1} \cdot \mathcal{I}_{\widetilde{W}}$. The weak transform $\widetilde{W}$ is given by the equations

$$\begin{aligned} uw - vz &= 0 \\ uz(\lambda u - z) &= 0 \\ uw(\lambda v - w) &= 0 \\ vw(\lambda u - z) &= 0 \\ vw(\lambda v - w) &= 0 \end{aligned}$$

We prove now that the weak transform $\widetilde{W}$ concides with the strict transform $\widehat{W}$. By proposition 3.3 and its proof we just have to show that the morphism $\lambda : \mathcal{O}_{\widetilde{W}}(-E_1) \longrightarrow \mathcal{O}_{\widetilde{W}}$ is injective. Now, $\widetilde{W}$ is contained in the hypersurface H of Y_1 defined by the equation $uw - vz = 0$. Over H we can globally write $z = \mu u$, $w = \mu v$, where μ can be seen as a section in $H^0(\mathcal{O}_H(E_1))$. Then $\widetilde{W}$ is given, inside H, by the equations

$$\begin{aligned} u^3\mu(\lambda-\mu) &= 0 \\ uv^2\mu(\lambda-\mu) &= 0 \\ uv^2\mu(\lambda-\mu) &= 0 \\ v^3\mu(\lambda-\mu) &= 0 \end{aligned}$$

Since u and v do not vanish at the same time, the weak transform is given by the equation $\mu(\lambda-\mu) = 0$ inside the hypersurface H, with respect to the coordinates $([u, v], \lambda, \mu)$. Hence λ is not zero divisor in $\widetilde{W}$ and $\widetilde{W} = \widehat{W}$. Hence

$$\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_1} \cdot \mathcal{I}_{\widehat{W}} \,.$$

Now $\widehat{W}$ is clearly the union, inside H, of the two smooth surfaces $\widehat{W_2}$ and $\widehat{W_3}$ intersecting transversally along a smooth curve inside the exceptional divisor E_1. Consider now the blow-ups $f_2 : \mathrm{Bl}_{\widehat{W}_2} Y_1 \longrightarrow Y_1$, with exceptional divisor E_2, and $f_3 : \mathrm{Bl}_{\widehat{\widehat{W}}_3} Y_2 \longrightarrow Y_2$, with exceptional divisor E_3; denote with $\widehat{\widehat{E_1}}$ and $\widehat{E_2}$ the strict transforms of E_1 and E_2 in Y_3, respectively. Let now $g := f_2 \circ f_3$ and let $f := f_1 \circ g$. Then by lemma 3.6 we have

$$f^{-1}(\mathcal{I}_W) = g^{-1}(\mathcal{I}_{f_1^{-1}(W)}) = g^{-1}(\mathcal{I}_{E_1}) \cdot g^{-1}(\mathcal{I}_{\widehat{W}}) = \mathcal{I}_{\widehat{\widehat{E_1}}} \cdot \mathcal{I}_{\widehat{E}_2} \cdot \mathcal{I}_{E_3} \,,$$

where we used that $\widetilde{E_1} = \widehat{E_1}$ and $\widetilde{\widehat{E_1}} = \widehat{\widehat{E_1}}$ by remark 3.4. Hence $f^{-1}(\mathcal{I}_W)$ is invertible and isomorphic to $\mathcal{O}_Y(-\widehat{\widehat{E_1}} - \widehat{E}_2 - E_3)$; it is now easy to show that the latter coincides with the anticanonical divisor $-K_Y$. □

As an immediate consequence of remarks 3.8, 3.9 and 3.10 we deduce the

Corollary 3.12. *The map $f : Y \longrightarrow V$ factors through a crepant resolution $h : Y \longrightarrow \mathrm{Bl}_W V$. Consequenty the map $h \times \mathrm{id} : Y \times \mathbb{C}^2 \longrightarrow \mathrm{Bl}_W V \times \mathbb{C}^2 \simeq B^3$ identifies to a crepant resolution of B^3.*

Let now X be an arbitrary smooth algebraic surface and let Δ_{I_1}, Δ_{I_2}, Δ_{I_3} be the pairwise diagonals Δ_I, $|I| = 2$, taken in whatever order. We have the following

Theorem 3.13. *The composition of blow-ups $s := s_1 \circ s_2 \circ s_3$*

$$Y := \mathrm{Bl}_{\widehat{\widehat{\Delta}}_{I_3}} Y_2 \xrightarrow{s_3} Y_2 := \mathrm{Bl}_{\widehat{\Delta}_{I_2}} Y_1 \xrightarrow{s_2} Y_1 := \mathrm{Bl}_{\Delta_{I_1}} X^3 \xrightarrow{s_1} X^3$$

where $\widehat{\Delta}_{I_2}$ and $\widehat{\widehat{\Delta}}_{I_3}$ are the strict transforms of Δ_{I_2} and Δ_{I_3} in Y_1 and Y_2, respectively, is a log-resolution of the pair $(X^3, \mathcal{I}_{\Delta_3})$ such that $s^{-1}(\mathcal{I}_{\Delta_3})$ is an invertible ideal isomorphic to the relative anticanonical $-K_{Y/X^3}$. Hence s factors through a crepant resolution $g : Y \longrightarrow B^3$ of the isospectral Hilbert scheme B^3.

Proof. Locally over X^3, the map s coincides precisely with $\varphi^{-1} \circ (f \times \mathrm{id}_{\mathbb{C}^2})$, where f is the birational map built in theorem 3.11 and φ is the map (2.1). The theorem is then an immediate consequence of proposition 3.11 and remarks 3.8 and 3.9. □

3.2 An $\mathfrak{S}_3$-equivariant resolution of B^3

Consider the 4-dimensional vector space $V = (\mathbb{C}^2)^2$ with coordinates (x, y, z, w) and the subscheme $W = W_1 \cup W_2 \cup W_3$ introduced in subsection 3.1. Consider the blow-up $f_1 : Y_1 := \mathrm{Bl}_0(V) \longrightarrow V$ of V at the origin and let E_0 be its exceptional divisor; since it can be identified with the total space of the Hopf line bundle over the projective space $\mathbb{P}(V)$, the variety Y_1 is equipped with a fibration $Y_1 \longrightarrow \mathbb{P}(V)$. Now, the polinomial $q = xw - yz$ defines a smooth quadric Q in $\mathbb{P}(V)$, which can be seen as a smooth subvariety of Y_1 inside E_0, thanks to the embedding of $\mathbb{P}(V)$ into Y_1 given by the zero section of the Hopf bundle.

Proposition 3.14. *The birational morphism $f : Y \longrightarrow V$ defined as the composition of smooth blow-ups*

$$Y = Y_3 \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} V$$

where $Y_2 = \mathrm{Bl}_{\widehat{W}}(Y_1)$, $Y_3 = \mathrm{Bl}_{\widehat{Q}}(Y_2)$, where $\widehat{W}$ and $\widehat{Q}$ are the strict transforms of W and Q in Y_1 and Y_2, respectively, is an isomorphism outside $f^{-1}(W)$. Moreover the ideal sheaf $f^{-1}(\mathcal{I}_W)$ is given by

$$f^{-1}(\mathcal{I}_W) = \mathcal{O}_Y(-2\widehat{\widehat{E_0}} - \widehat{E_{\widehat{W}}} - 3E_{\widehat{Q}})$$

where $E_{\widehat{W}}$ and $E_{\widehat{Q}}$ are the exceptional divisors in Y_2 and Y_3, respectively, and where $\widehat{E_{\widehat{W}}}$ and $\widehat{\widehat{E_0}}$ are the strict transforms of $E_{\widehat{W}}$ and E_0 in Y.

Proof. Since $\mathrm{ord}_0 \mathcal{I}_W = 2$, we have

$$\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_0}^2 \cdot \mathcal{I}_{\widetilde{W}}$$

where $\widetilde{W}$ is the weak transform of W in Y_1. By a computation in coordinates, using the same generators for $\mathcal{I}_W$ we used in the proof of theorem 3.11, one gets

$$\mathcal{I}_{\widetilde{W}} = \mathcal{I}_Q \cap \mathcal{I}_{\widehat{W}} \ ,$$

that is, the weak transform $\widetilde{W}$ is the scheme-theoretic union of the quadric Q and the strict transform $\widehat{W}$ of W in Y_1, which is a smooth codimension 2 subvariety with three irreducible components $\widehat{W}_i$, $i = 1, \ldots, 3$. Moreover $\widehat{W} \cap E_0$ is contained in Q and is precisely the union of three skew lines in $E_0 \simeq \mathbb{P}(V)$; hence $\widehat{W} \cap E_0$ is a smooth codimension 3 subvariety of Y_1. Therefore the hypothesis of lemma 3.7 are satisfied; this means that, when blowing up the strict transform $\widehat{W}$ in Y_1 one gets

$$f_2^{-1}(\mathcal{I}_Q \cap \mathcal{I}_{\widehat{W}}) = \mathcal{I}_{E_{\widehat{W}}} \cdot \mathcal{I}_{\widehat{Q}} \ .$$

Since $\mathrm{ord}_{\widehat{W}} \widehat{E_0} = 0$, we get

$$(f_1 \circ f_2)^{-1}(\mathcal{I}_W) = \mathcal{I}_{\widehat{E_0}}^2 \cdot \mathcal{I}_{E_{\widehat{W}}} \cdot \mathcal{I}_{\widehat{Q}} \ .$$

Remembering that $\mathrm{ord}_{\widehat{Q}} \widehat{E_0} = 1$, the last blow-up now yields the formula in the statement. □

Corollary 3.15. *The map $f : Y \longrightarrow V$ factors through a resolution $h : Y \longrightarrow \mathrm{Bl}_W V$. Consequenty the map $h \times \mathrm{id} : Y \times \mathbb{C}^2 \longrightarrow \mathrm{Bl}_W V \times \mathbb{C}^2 \simeq B^3$ identifies to an $\mathfrak{S}_3$-equivariant resolution of B^3.*

Consider now the case of an arbitrary smooth algebraic surface X. Consider the blow-up $s_1 : Y_1 := \mathrm{Bl}_{\Delta_{123}} X^3 \longrightarrow X^3$ of the small diagonal Δ_{123} in X^3 and let E_0 be its exceptional divisor. The situation is locally, over X^3, analogous to the one just studied. Hence it is now clear that $s_1^{-1}(\mathcal{I}_{\Delta_3}) = \mathcal{I}_{E_0}^2 \cdot (\mathcal{I}_Q \cap \mathcal{I}_{\widehat{\Delta_3}})$, where $\widehat{\Delta_3}$ is the strict transform of Δ_3 in Y_1 and where Q is a quadric subbundle of $\mathbb{P}(N_{\Delta_{123}/X^3})$ over Δ_{123} and hence a smooth subvariety of Y_1 inside E_0. We have the following theorem

Theorem 3.16. *The composition of smooth blow-ups $s := s_1 \circ s_2 \circ s_3$:*

$$Y := \mathrm{Bl}_{\widehat{Q}} Y_2 \xrightarrow{s_3} Y_2 := \mathrm{Bl}_{\widehat{\Delta_3}} Y_1 \xrightarrow{s_2} Y_1 \xrightarrow{s_1} X^3$$

where $\widehat{\Delta}_3$ and $\widehat{Q}$ are the strict transforms of Δ_3 and Q in Y_1 and Y_2, respectively, defines a $\mathfrak{S}_3$-equivariant log-resolution of the pair $(X^3, \mathcal{I}_{\Delta_3})$ and hence factors through a $\mathfrak{S}_3$-equivariant log-resolution $g : Y \longrightarrow B^3$ of the isospectral Hilbert scheme B^3.

Proof. The map s is clearly $\mathfrak{S}_3$-equivariant and, locally over X^3, coincides with the map $\varphi^{-1} \circ (f \times \mathrm{id}_{\mathbb{C}^2})$, where f is the map introduced in proposition 3.14 and where φ is the map (2.1). The content of the theorem is then a consequence of proposition 3.14, corollary 3.15 and remarks 3.8 and 3.9. □

Remark 3.17. This resolution is not crepant, as one gets easily $K_{Y/X^3} + s^{-1}(\mathcal{I}_{\Delta_3}) = \mathcal{O}(\widehat{\widehat{E_0}} + E_{\widehat{Q}})$, where $E_{\widehat{Q}}$ is the exceptional divisor in Y_3 and where $\widehat{\widehat{E_0}}$ is the strict transform of E_0 in Y.

Remark 3.18. The step Y_2 coincides with the Fulton-MacPherson compactification $X[3]$ of $X^3 \setminus \Delta_3$ (see [FM94]).

Remark 3.19. By construction, the resolution Y is equipped with a $\mathfrak{S}_3$-action. The stabilizer of any point for this action is trivial. Hence, passing to the quotient modulo $\mathfrak{S}_3$, the induced map $\hat{f} : Y/\mathfrak{S}_3 \longrightarrow S^3X$ provides an explicit resolution of S^3X which factors through the Hilbert scheme of points $X^{[3]} = B^3/\mathfrak{S}_3$.

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