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# On realizations of Dirac structures by closed two-forms

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In this short note we prove that any Dirac structure can be realized as the push-forward of a presymplectic form via a surjective submersion, and moreover, that this submersion fits into a full dual pair of Dirac structures, a notion we introduce here, building upon those of *pre-dual pairs* and *presymplectic realizations*. The construction subsumes the symplectic realization of Crainic et al. [2011], but the proof given here is completely conceptual, revealing the Dirac geometric nature of the problem. In the second part of the paper, we discuss the general problem of pushing forward Dirac structures under submersions, proving a Dirac version of Libermann's theorem Libermann [1983], with special attention paid to the case of closed two-forms.

## 1 Introduction

Given a Dirac structure $L \subset TM \oplus T^*M$ on a manifold $M$, one seeks a surjective submersion $\mathbf{s} : \Sigma \to M$ and a closed two-form $\omega$ on $\Sigma$, such that

$$\mathbf{s} : (\Sigma, \omega) \longrightarrow (M, L)$$

is a forward Dirac map (we refer the reader to e.g. Bursztyn [2013] for background material on Dirac structures). This is one possible Dirac-theoretic incarnation of the construction of *symplectic* realizations of Poisson manifolds, the importance of which had been manifest since the early days of Poisson geometry (cf. Weinstein [1983] for the local construction, and Coste et al. [1987] for the gluing argument and the ensuing global realization). A global direct proof of the existence of symplectic realizations was presented in Crainic et al. [2011], and our main result is a natural extension of this construction to the Dirac setting. However, the inconvenient feature of the construction in Crainic et al. [2011] is that, albeit its formulation was crystal-clear and completely conceptual, its *proof* was in some sense rather artificial. The solution presented here, of the more

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general problem for Dirac manifolds, is completely conceptual, as it relies on natural Dirac geometric operators (like pullback, gauge transformations), which are not generally available in the Poisson stetting.

### The main theorem

We introduce the ingredients needed in order to state our main result. A **spray** for a Dirac structure $L \subset TM \oplus T^*M$ is a vector field $\mathcal{V} \in \mathfrak{X}(L)$, satisfying:

1. $\mathbf{s}_* \mathcal{V}_a = \mathrm{pr}_{TM}(a)$, for all $a \in L$;
2. $m_t^* \mathcal{V} = t\mathcal{V}$, where $m_t : L \to L$ denotes multiplication by $t \neq 0$.

For example, given a linear connection on $L$ with horizontal lift $h$, the vector field $\mathcal{V}_a := h_a(\mathrm{pr}_{TM}(a))$ is a spray on $L$. The **two-form** on $L$ is:

$$\omega_L = \mathrm{pr}^*_{T^*M}\omega_{\mathrm{can}} \in \Omega^2(L),$$

where $\omega_{\mathrm{can}}$ is the canonical two-form on $T^*M$.

We prove the following:

**Theorem 1.1.** Let $L$ be a Dirac structure on $M$, and denote by $\mathbf{s} : L \to M$ the bundle projection. Let $\mathcal{V} \in \mathfrak{X}(L)$ be a spray for $L$ and denote its local flow by $\varphi_\epsilon : L \xrightarrow{\sim} L$, which is defined on some open neighborhood of $M$ in $L$. Define:

$$\omega := \int_0^1 \varphi_\epsilon^* \omega_L \mathrm{d}\epsilon, \quad \mathbf{t} := \mathbf{s} \circ \varphi_1.$$

Then, on some open neighborhood $\Sigma$ of $M$ in $L$, we have that:

(i) The following **robustness** condition (cf. Bursztyn et al. [2004]) holds:

$$\ker \mathbf{s}_* \cap \ker \omega \cap \ker \mathbf{t}_* = 0.$$

(ii) The following **dual pair relation** holds:

$$\mathbf{s}^* L^{-\omega} = \mathbf{t}^* L.$$

(iii) We have a diagram of **forward Dirac** maps:

$$(M, L) \xleftarrow{\mathbf{s}} (\Sigma, \omega) \xrightarrow{\mathbf{t}} (M, -L).$$

□

In the Poisson setting, this reduces to the formula in Crainic et al. [2011], which has striking applications for normal forms in Poisson geometry; a fact we have exploited rather systematically in Frejlich et al. [2013, 2015].

### The origin of the formula

The formula originates in the path-space approach to integrability of Lie algebroids, developed in Crainic et al. [2003]. In a nutshell, given a Lie algebroid $A$ over a manifold $M$, the space $P(A)$ of $A$-paths carries a canonical *homotopy foliation*, of finite codimension, whose leaf space has a canonical structure of topological groupoid $G(A) \rightrightarrows M$ (the so-called *Weinstein groupoid* of $A$), which is smooth exactly when $A$ is integrable by a Lie groupoid. If the Lie algebroid is a Dirac $A = L$ on $M$, and moreover, if it is integrable, then $G_L$ carries a canonical, multiplicative, closed two-form $\omega_{G(L)}$, for which the source and target maps $\mathbf{s}, \mathbf{t} : (G(L), \omega_{G(L)}) \to (M, L)$ satisfy the conclusion of our main theorem (see Bursztyn et al. [2004]). However, even if $L$ is not integrable, the Banach manifold $P(L)$ of $L$-paths carries a canonical two-form $\omega_{P(L)}$ which is basic for the homotopy foliation, and, in the integrable case, it is the pullback of $\omega_{G(L)}$. A spray on $L$ induces an exponential map $\exp_{\mathcal{V}} : \Sigma \to P(L)$ on a neighborhood $\Sigma \subset L$ of the zero-section, which is transverse to the homotopy foliation. The two-form $\omega$ from our main result is the precisely the pullback of $\omega_{P(L)}$ via $\exp_{\mathcal{V}}$ (c.f. the explicit formula of the $\omega_{P(L)}$ from [Bursztyn et al., 2004, Section 5]).

Along these lines, the machinery of presymplectic Lie groupoids from Bursztyn et al. [2004] can be used to give a different proof of our main result in the case when $L$ is integrable; however, such a proof is bound to be less elementary (and less general).

### Pushing forward Dirac structures

In the last two sections of the paper, we discuss the Dirac-theoretic machinery underlying our main theorem; namely, the problem of pushing forward Dirac structures under a surjective submersion. At the heart of the discussion lies *Libermann's theorem* Libermann [1983], which states that a surjective submersion $\mu : \Sigma \to M$ with connected fibers pushes forward a symplectic form $\omega \in \Omega^2(\Sigma)$ to a Poisson structure on $M$ if, and only if, the symplectic orthogonal $V^{\perp}$ to the vertical foliation $V := \ker \mu_*$ is involutive, that is, if it defines a foliation. The general Dirac version of this result reads:

**Proposition 1.2** (Dirac-Libermann, general version)**.** A Dirac structure $L$ on $\Sigma$ can be pushed forward via $\mu$ to a Dirac structure on $M$ if, and only if $L_p^{\mu} := \mu^* \mu_* L_p \subset T_p\Sigma \oplus T_p^*\Sigma$ is a Dirac structure on $\Sigma$. □

We turn next to the case where the Dirac structure $L$ corresponds to a closed two-form on $\Sigma$. In this case, the above proposition reduces to:

**Proposition 1.3** (Dirac-Libermann, case of two-forms)**.** A closed two-form $\omega$ can be pushed forward via $\mu$ to a Dirac structure on $M$ if, and only if $V + (V^{\perp})^{\omega}$ is a smooth bundle and $V^{\perp}$ is involutive. □

Criteria for this to hold are discussed, and they lead naturally to a Dirac version of *dual pairs*, which we introduce here, and which is closely related to the notions of *presymplectic realizations* of Bursztyn et al. [2004] and *pre-dual pairs* of Bursztyn et al. [2003]. We conclude the paper by examining the relations between these objects.

## 2 Proof of the main Theorem

**P**roof of (i). Since the robustness condition is open, it suffices to check it along the zero section $M \subset L$. On $M$ we have a canonical splitting $TL|_M = TM \oplus L$. Since $\mathcal{V}$ vanishes on $M$, the spray condition implies that the differential of its flow along $M$ has the following form:

$$\varphi_{\epsilon *}(u, v + \eta) = (u + \epsilon v, v + \eta), \quad (u, v + \eta) \in T_x M \oplus L_x. \tag{1}$$

On the other hand, the canonical two-form $\omega_L$ has the following form along the zero section:

$$\omega_L\left((u, v + \eta), (u', v' + \eta')\right) = \eta'(u) - \eta(u').$$

Thus, we have that:

$$\begin{aligned} \omega\left((u, v + \eta), (u', v' + \eta')\right) &= \int_0^1 \omega_L\left(\varphi_{\epsilon *}(u, v + \eta), \varphi_{\epsilon *}(u', v' + \eta')\right) \mathrm{d}\epsilon = \\ &= \eta'(u + 1/2v) - \eta(u' + 1/2v'). \end{aligned} \tag{2}$$

Let $(u, v + \eta) \in (\ker \mathbf{s}_* \cap \ker \mathbf{t}_* \cap \ker \omega)\,|_M$. Since:

$$\mathbf{s}_*(u, v + \eta) = u, \quad \mathbf{t}_*(u, v + \eta) = \mathbf{s}_* \varphi_{1*}(u, v + \eta) = \mathbf{s}_*(u + v, v + \eta) = u + v,$$

it follows that $u = 0$ and $v = 0$. By formula (2),

$$0 = \omega((u, v + \eta), (u', 0)) = -\eta(u')$$

for all $u'$; thus $\eta = 0$. This finishes the proof. ■

**P**roof of (ii). Let $\lambda_L := \mathrm{pr}^*_{T^*M} \lambda_{\mathrm{can}} \in \Omega^1(L)$ be the pullback of the tautological one-form $\lambda_{\mathrm{can}}$ on $T^*M$. Then $\mathrm{d}\lambda_L = -\omega_L$. Note that condition (1) in the definition of a spray $\mathcal{V}$ is equivalent to $\mathcal{V} + \lambda_L$ being a section of $\mathbf{s}^* L$. The local flow $\Phi_\epsilon$ of the section $\mathcal{V} + \lambda_L$ on $\mathbf{s}^* L$ covers $\varphi_\epsilon : L \to L$ and, on its domain of definition, it is given by (e.g. [Gualtieri, 2011, Proposition 2.3]):

$$\Phi_\epsilon : \mathbf{s}^* L \longrightarrow \mathbf{s}^* L, \quad \Phi_\epsilon(u + \eta) = \varphi_{\epsilon *}(u) + \varphi_\epsilon^{* -1}(\iota_u B_\epsilon + \eta),$$

where $B_\epsilon = \int_0^\epsilon \varphi_s^* \mathrm{d}\lambda_L \mathrm{d}s = -\int_0^\epsilon \varphi_s^* \omega_L \mathrm{d}s$. Let $\Sigma \subset L$ be an open neighborhood of $M$ on which $\varphi_\epsilon$ is defined up to time one. Since $B_1 = -\omega$, we have that:

$$\mathbf{s}^* L|_{\varphi_1(\Sigma)} = \Phi_1 \left(\mathbf{s}^* L|_\Sigma\right) = \varphi_{1*}(\mathbf{s}^* L^{-\omega}|_\Sigma),$$

and therefore

$$\mathbf{s}^* L^{-\omega}|_\Sigma = \varphi_1^*(\mathbf{s}^* L|_{\varphi_1(\Sigma)}) = \mathbf{t}^* L|_\Sigma. \qquad \blacksquare$$

**P**roof of (iii). Let $L_\omega \subset T\Sigma \oplus T^*\Sigma$ denote the graph of $\omega$. We will prove that

$$\mathbf{s} : (\Sigma, L_\omega) \to (M, L)$$

is a forward Dirac map; the second part follows similarly. Let $p \in \Sigma$ and $x := s(p)$. We will show that $\mathbf{s}_*(L_{\omega,p}) = L_x$. Let $u \in (\ker \mathbf{t}_*)_p$. Then

$$u + \iota_u \omega \in \mathbf{t}^*(L)^\omega = \mathbf{s}^*(L).$$

This implies that $\iota_u \omega = \mathbf{s}^* \xi_u$, for some $\xi_u \in T_x^* M$, and that $\mathbf{s}_* u + \xi_u \in L_x$. But clearly, also $\mathbf{s}_* u + \xi_u \in \mathbf{s}_*(L_{\omega,p})$. Note that by (i) the map $u \mapsto \mathbf{s}_* u + \xi_u$ is injective, so its image will be a subspace of dimension $\dim(M)$ in $L_x \cap \mathbf{s}_*(L_{\omega,p})$. Thus $L_x = \mathbf{s}_*(L_{\omega,p})$. $\blacksquare$

## 3 Pushing forward Dirac structures

In this section, we discuss the problem of pushing forward a closed two-form as a Dirac structure through a surjective submersion. First, we note the following Dirac-geometric version of Libermann's theorem:

**Proposition 3.1** (Dirac-Libermann, general version)**.** Let $\mu : \Sigma \to M$ be a surjective submersion with connected fibres. A Dirac structure $L$ on $\Sigma$ can be pushed forward via $\mu$ to a Dirac structure on $M$ if, and only if:

$$L_p^\mu := \mu^* \mu_* L_p \subset T_p\Sigma \oplus T_p^*\Sigma$$

is a Dirac structure on $\Sigma$. $\square$

**Remark 1.** Note that $p \mapsto L_p^\mu$ is a family of Lagrangian subspaces, which in general is not smooth. A sufficient condition ensuring smoothness of $L^\mu$ is that $\mathrm{pr}_{T^*\Sigma} L + \mu^*(T^*M) = T^*\Sigma$. $\square$

**Example 3.2.** As an example where $L^\mu$ fails to be smooth, consider $\mu : \mathbb{R}^2 \to \mathbb{R}$, $\mu(x,y) = x$, and $L$ the Dirac structure on $\mathbb{R}^2$ corresponding to the closed two-form $\omega = x\mathrm{d}x \wedge \mathrm{d}y$. In this case, for $x \neq 0$, $L_{(x,y)}^\mu = \langle \frac{\partial}{\partial y}, \mathrm{d}x \rangle$, and yet, $L_{(0,y)}^\mu = T_{(0,y)}\mathbb{R}^2$. Moreover, these spaces lie even in different connected components of the space of Lagragian subspaces of $T\mathbb{R}^2 \oplus T^*\mathbb{R}^2$. $\square$

The proof of Proposition 3.1 makes use the following characterization of "basic" Dirac structures:

**Proposition 3.3** (Basic Dirac structures)**.** Let $\mu : \Sigma \to M$ be a surjective submersion with connected fibres, and denote by $V := \ker \mu_* \subset T\Sigma$. Then a Dirac structure $L$ on $\Sigma$ is the pullback $L = \mu^* L_M$ of a Dirac structure $L_M$ on $M$ if, and only if, $V \subset L$. In this case, we also have that $\mu_* L = L_M$, i.e. $\mu : (\Sigma, L) \to (M, L_M)$ is a forward Dirac map. □

**P**roof of Proposition 3.3. If $L = \mu^* L_M$, then clearly $V \subset L$ and, since $\mu$ is a submersion, we also have that $\mu_* \mu^* L_M = L_M$.

Conversely, assume that $V \subset L$. This implies that the flow of vector fields in $V$ preserves $L$. Since the fibres of $\mu$ are connected, this implies that, for every $p, q \in \mu^{-1}(x)$, we can find a diffeomorphism $\varphi : \Sigma \to \Sigma$ so that $\varphi(p) = q$, $\varphi_* L_p = L_q$, and which is vertical: $\mu \circ \varphi = \mu$. So $\mu_* L_q = \mu_* \varphi_* L_p = \mu_* L_p$. Thus, there is a well-defined family of Lagrangian subspaces $x \mapsto L_{M,x} \in T_x M \oplus T^*_x M$, such that $\mu_* L_p = L_{M,\mu(p)}$ for all $p \in \Sigma$. Smoothness and integrability of $L_M$ are proven as follows. First, remark that the submersion $\mu$ admits local sections $\sigma : U \to \Sigma$, $U \subset M$, and that any such local section $\sigma$ is transverse to $L$, i.e. it satisfies:

$$\sigma_*(T_x M) + \mathrm{pr}_{T\Sigma}(L_{\sigma(x)}) = T_{\sigma(x)}\Sigma, \quad \forall\, x \in U$$

and this condition implies that $\sigma^* L$ is a smooth Dirac structure on $U$ (see e.g. Bursztyn [2013]). Second, note that $V \subset L$ implies that $L \subset T\Sigma \oplus \operatorname{im} \mu^*$. Now, if $v + \sigma^* \alpha \in \sigma^* L$, then $\sigma_* v + \alpha \in L \subset T\Sigma \oplus \operatorname{im} \sigma^*$, and so $\alpha = \mu^* \beta$ for some $\beta \in T^* M$; hence $\mu_* \sigma_* v + \beta = v + \beta \in L_M$. Thus, $\sigma^* L \subset L_M|_U$; counting dimensions we conclude that: $L_M|_U = \sigma^* L$. This proves that $L_M$ is a smooth Dirac structure. Let us conclude by showing that $\mu^* L_M = L$. An element in $\mu^* L_M$ has the form $v + \mu^* \alpha$, where $\mu_* v + \alpha \in L_M$. Since $L_M = \mu_* L$, there exists $w \in T\Sigma$ so that $\mu_* w = \mu_* v$ and $w + \mu^* \alpha \in L$. But then $w - v \in V \subset L$; hence $v + \mu^* \alpha = (w + \mu^* \alpha) + (v - w) \in L$. ■

**P**roof of Proposition 3.1. First, if $L$ can be pushed forward to a Dirac structure $L_M$ on $M$, then $L^\mu = \mu^* L_M$ is a Dirac structure on $\Sigma$, as it is the pullback of a Dirac structure through a surjective submersion.

Conversely, assume that $L^\mu$ is a Dirac structure. Note that $\mu_* L^\mu_p = \mu_* L_p$ for all $p \in \Sigma$; hence it suffices to check that $L^\mu$ can be pushed forward. Since $V \subset L^\mu$, this follows by Proposition 3.3. ■

As an example, consider the case when $L$ is a *coupling Dirac structure* (also called *horizontally nondegenerate Dirac structure*), i.e. a Dirac structure $L$ satisfying:

$$L \cap (V \oplus V^\circ) = 0,$$

where $V := \ker \mu_*$, and $V^\circ \subset T^*\Sigma$ denotes the annihilator of $V$. A Lagrangian $L \subset T\Sigma \oplus T^*\Sigma$ satisfying the condition above can be described by a Vorobjev triple $(H, \omega, \pi)$, where $H \subset T\Sigma$ is a complement of $V$, $\omega$ is a

2-form on $H$, and $\pi$ is a vertical bivector:

$$T\Sigma = H \oplus V, \quad \omega \in \Gamma(\wedge^2 V^\circ) \subset \Omega^2(\Sigma), \quad \pi \in \Gamma(\wedge^2 V) \subset \mathfrak{X}^2(\Sigma),$$

and $L$ has the direct sum decomposition:

$$L = \{v + \iota_v\omega \ : \ v \in H\} \oplus \{\xi + \pi^\sharp\xi \ : \ \xi \in H^\circ\}.$$

It corresponds to a Dirac structure if, and only if, the following conditions are satisfied:

(a) $\pi$ is a Poisson structure: $[\pi, \pi] = 0$,

(b) $\mathscr{L}_{u_1}\pi \in \Gamma(H \wedge T\Sigma)$,

(c) $[u_1, u_2] + \pi^\sharp\iota_{u_1}\iota_{u_2}\mathrm{d}\omega \in \Gamma(H)$,

(d) $\mathrm{d}\omega(u_1, u_2, u_3) = 0$,

for all $u_1, u_2, u_3 \in \Gamma(H)$. For these general properties of coupling Dirac structures, see e.g. Wade [2008].

In this situation, Proposition 3.1 specializes to:

**Corollary 3.4** (Coupling Dirac structures)**.** Let $\mu : \Sigma \to M$ be a surjective submersion with connected fibers, and let $L$ be a coupling Dirac structure on $\Sigma$, with corresponding Vorobjev triple $(H, \omega, \pi)$. Then $L$ can be pushed forward to a Dirac structure on $M$ if and only if $\omega$ is closed. In this case, $\omega = \mu^*\eta$, where $\eta$ is a closed two-form on $M$, $\mu : (\Sigma, L) \to (M, \eta)$ is a forward Dirac map, and moreover, $H$ is an involutive distribution. □

**Proof**. It is easy to check that $L^\mu$ is the graph of $\omega$. This is always a smooth bundle, and it is a Dirac structure if and only if $\omega$ is closed. Thus, the first part follows from Proposition 3.1. By Proposition 3.3, $\omega$ is basic as a Dirac structure, and hence it is basic as a two-form, i.e. $\omega = \mu^*\eta$, where $\eta$ is a closed two-form on $M$, and $\mu : (\Sigma, L) \to (M, \eta)$ is a forward Dirac map. By (c) above, $\mathrm{d}\omega = 0$ implies that $H$ is involutive. ■

## 4 Pushing forward closed two-forms

In this section, we specialize the discussion to the case of closed two-forms. Let us fix a surjective submersion $\mu : \Sigma \to M$ with connected fibers, and a closed two-form $\omega$ on $\Sigma$. Throughout the section, we will use the following notation:

$$V := \ker \mu_*, \qquad K := \ker \omega,$$
$$V^\perp := \{u \in T\Sigma \ : \ \omega(v, u) = 0 \ \forall \ v \in V\},$$
$$(V^\perp)^\omega := \{u + \iota_u\omega \ : \ u \in V^\perp\}.$$

The family of Lagrangian subspaces from Proposition 3.1 becomes:

$$L^{\mu}_{\omega} = \mu^{*}\mu_{*}L_{\omega} = V + (V^{\perp})^{\omega},$$

where $L_{\omega}$ denotes the graph of $\omega$. Therefore, we obtain:

**Corollary 4.1.** The closed two-form $\omega$ can be pushed forward to a Dirac structure on $M$ if, and only if:

$$V + (V^{\perp})^{\omega} \subset T\Sigma \oplus T^{*}\Sigma$$

is a Dirac structure on $\Sigma$. Moreover, if this holds, then the induced Dirac structure on $M$ is Poisson if, and only if:

$$K \subset V.$$

□

**Proof**. Proposition 3.1 implies the first part. For the second part, note that:

$$\mu_{*}L_{\omega,p} \cap T_{\mu(p)}M = \{\mu_{*}(v) \ : \ v \in K_p\};$$

therefore the induced Dirac structure is Poisson if, and only if, this space is trivial, which is equivalent to $K \subset V$. ■

For verifying conditions that are closed, we will use the following:

**Lemma 4.2.** There is an open and dense subset $U \subset \Sigma$ on each of whose connected components $K$, $V \cap K$ and $V^{\perp}$ are smooth distributions. □

**Proof**. First, the set $U_0$ consisting of regular points of $\omega$, i.e. where $\omega$ has locally constant rank, is open and dense. Thus, on the connected components of $U_0$, $K := \ker \omega$ is a smooth distribution on $\Sigma$. Also the $U \subset U_0$, consisting of points where $V \cap K$ has locally constant rank is open and dense. On each connected component of $U$, $\omega^{\sharp}(V) \subset T^{*}\Sigma$ is the image of a constant rank vector bundle map; therefore its annihilator $V^{\perp} \subset T\Sigma$ has constant rank. ■

We can now prove a version of Corollary 4.1, which is closer to the classical Theorem of Libermann:

**Proposition 4.3** (Dirac-Libermann, case of two-forms)**.** Assume that $V + (V^{\perp})^{\omega}$ is a smooth bundle. Then $V + (V^{\perp})^{\omega}$ is a Dirac structure if and only if $V^{\perp}$ is involutive, in the sense that the Lie bracket of any two smooth vector fields in $V^{\perp}$ lies in $V^{\perp}$. □

**Remark 2.** Note that $V^\perp$ need not be a singular distribution (in the sense of Stefan [1974]), simply because its rank need not be lower semicontinuous. For instance, in Example 3.2, we have:

$$V^\perp_{(x,y)} = \begin{cases} V_{(x,y)} & \text{when } x \neq 0; \\ T_{(x,y)}\Sigma & \text{when } x = 0. \end{cases}$$

The same problem can arise even if $\mu$ is a forward Dirac map (so $V + (V^\perp)^\omega$ is smooth and $V^\perp$ is involutive); see Example 4.5. □

**P**roof of Proposition 4.3. Recall Bursztyn [2013] that for a smooth Lagrangian distribution $L$, integrability is equivalent to the vanishing of the three tensor:

$$\Upsilon \in \Gamma\big(\wedge^3 L^*\big), \quad \Upsilon(s_1, s_2, s_3) = \langle [s_1, s_2], s_3 \rangle,$$

where $[\cdot, \cdot]$ denotes the Dorfman bracket.

Denote $V + (V^\perp)^\omega$ by $L$. Note that if $u_1, u_2 \in \mathfrak{X}(\Sigma)$ are two vector fields, such that $u_{1,p}, u_{2,p} \in V_p^\perp$ for all $p \in \Sigma$, then, for any $v \in \Gamma(V)$, we have that:

$$\Upsilon(u_1, u_2, v) = \langle [u_1 + \iota_{u_1}\omega, u_2 + \iota_{u_2}\omega], v \rangle = \langle [u_1, u_2] + \iota_{[u_1,u_2]}\omega, v \rangle = \omega([u_1, u_2], v),$$

where we have used the fact that $\omega$ is closed. Hence the vanishing of $\Upsilon$ implies that $[u_1, u_2] \in V^\perp$. Thus $L$ Dirac implies $V^\perp$ involutive.

Conversely, assume that $V^\perp$ is involutive. First, we check that $\Upsilon$ vanishes on sections $s \in \Gamma(L)$ of the forms (i) $s = v$, where $v \in \Gamma(V)$, and (ii) $s = u + \iota_u\omega$, where $u$ is a smooth vector field that takes values in $V^\perp$. For $i = 1, 2, 3$, let $v_i$ be a section of the first type, and $u_i + \iota_{u_i}\omega$ be a section of the second type. Clearly, $\Upsilon(v_1, v_2, v_3) = 0$;

$$\Upsilon(v_1, v_2, u_1 + \iota_{u_1}\omega) = \omega([v_1, v_2], u_1) = 0,$$

because $V$ is involutive;

$$\Upsilon(u_1 + \iota_{u_1}\omega, u_2 + \iota_{u_2}\omega, u_3 + \iota_{u_3}\omega) = 0,$$

because $\omega$ is closed and the three elements belong to the Dirac structure $L_\omega$. Finally,

$$\Upsilon(u_1 + \iota_{u_1}\omega, u_2 + \iota_{u_2}\omega, v) = \omega([u_1, u_2], v) = 0,$$

because $V^\perp$ is involutive.

On the other hand, by Lemma 4.2, there is an open and dense set $U$ so that $V^{\perp}$ is a smooth bundle on each connected component of $U$. At points $p \in U$, every element in $s \in L_p$ can be represented as $s = v_p + u_p + \iota_{u_p}\omega$, for some smooth vector fields $v$ and $u$ on $\Sigma$ (whose support is compact and included in $U$), and so that $v_q \in V_q$ and $u_q \in V_q^{\perp}$, for all $q \in \Sigma$. Thus, $\Upsilon = 0$ (since it vanishes on the dense set $U$), and therefore $L$ is integrable. ■

**Remark 3.** In notation of the proof above, note that, even if $L$ is a Dirac structure, not every element $s \in L_p$ can be represented as $s = v_p + u_p + \iota_{u_p}\omega$, for two smooth vector fields $v$ and $u$ so that $v \in V$ and $u \in V^{\perp}$; for instance, consider the element $s = \frac{\partial}{\partial x} \in L_0$ in Example 4.5 below. □

Next, we discuss a natural condition which implies that $\omega$ can be pushed forward to a Dirac structure on $M$. First, note that the following is a short exact sequence:

$$0 \longrightarrow V \cap K \longrightarrow V^{\perp} \longrightarrow \frac{V + (V^{\perp})^{\omega}}{V} \longrightarrow 0,$$

where the first map is the inclusion and the second is $u \mapsto [u + \iota_u\omega]$. Therefore, the following spaces are canonically isomorphic:

$$\frac{V^{\perp}}{V \cap K} \cong \frac{V + (V^{\perp})^{\omega}}{V}.$$

Thus, if $V + (V^{\perp})^{\omega}$ is a smooth vector bundle, then $\frac{V^{\perp}}{V \cap K}$ has a canonical vector bundle structure. Motivated by this discussion, we give a sufficient condition for smoothness and integrability.

**Proposition 4.4** (Push-forward criterion)**.** If there exists a smooth distribution $W \subset T\Sigma$, such that:

$$V^{\perp} = W + V \cap K, \tag{3}$$

then $V + (V^{\perp})^{\omega}$ is smooth.

If $W$ is also involutive, then $V + (V^{\perp})^{\omega}$ is a Dirac structure; hence, there is a Dirac structure $L_M$ on $M$ so that $\mu : (\Sigma, \omega) \to (M, L_M)$ is a forward Dirac map. □

**Proof**. Denote $L := V + (V^{\perp})^{\omega}$. By (3), $L$ is the image of the vector bundle map:

$$V \oplus W \longrightarrow T\Sigma \oplus T^*\Sigma, \quad (v, w) \mapsto v + w + \iota_w\omega. \tag{4}$$

Now, $L$ is a family of Lagrangian subspaces; in particular, it has constant rank. We conclude that $L$ is the image of a smooth vector bundle map of constant rank; therefore, $L$ is smooth, and this proves the first part.

Assume that $W$ is involutive. By using a smooth splitting of the vector bundle map (4), we can represent any section $s \in \Gamma(L)$ as $s = v + w + \iota_w\omega$, where $v \in \Gamma(V)$ and $w \in \Gamma(W)$. Using this, the proof that $L$ is involutive proceeds exactly as the second part of the proof of Proposition 4.3; namely, one shows that the tensor $\Upsilon$ vanishes on any three sections of the forms (i) $s = v$, where $v \in \Gamma(V)$, or (ii) $s = w + \iota_w\omega$, where $w \in \Gamma(W)$. ■

In general, a subbundle $W \subset T\Sigma$ satisfying (3) does not exist, even if $L_\omega$ can be pushed forward to a Dirac structure on $M$ (i.e. if $V + (V^\perp)^\omega$ is a Dirac structure), as the following example shows:

**Example 4.5.** Let $\Sigma := \mathbb{R}^3$, $M := \mathbb{R}$, $\mu(x, y, z) := x$ and

$$\omega = \mathrm{d}(x^2 y) \wedge \mathrm{d}z = x(2y\mathrm{d}x + x\mathrm{d}y) \wedge \mathrm{d}z.$$

Then:

$$\mu : (\mathbb{R}^3, \omega) \longrightarrow (\mathbb{R}, T\mathbb{R})$$

is a forward Dirac map. Indeed, consider the vector field $v := x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$. It satisfies $\iota_v \omega = 0$ and $\mu_*(v) = x\frac{\partial}{\partial x}$, which shows that for $x \neq 0$, we have $\mu_*(L_{\omega,(x,y,z)}) = T_x\mathbb{R}$. But this equality clearly extends over $x = 0$, since $\omega_{(0,y,z)} = 0$, proving that $\mu$ is a forward Dirac submersion.

Note also that $V = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$; at $x \neq 0$,

$$K = V^\perp = \langle v \rangle, \qquad K \cap V = 0$$

and at $x = 0$,

$$K = V^\perp = T_{(0,y,z)}\mathbb{R}^3, \quad K \cap V = V.$$

The regularity assumption (3) is not satisfied in this example; to see this, note that a complement $W$ would have to coincide with $\langle v \rangle$ on $x \neq 0$, and yet this cannot extend smoothly (as a line bundle) over points of the form $(0, 0, z)$, since:

$$\lim_{y \to 0} \lim_{x \to 0} \langle v \rangle = \left\langle \frac{\partial}{\partial y} \right\rangle \text{ and } \lim_{x \to 0} \lim_{y \to 0} \langle v \rangle = \left\langle \frac{\partial}{\partial x} \right\rangle.$$

□

Next, we note that the condition (3) is symmetric in $V$ and $W$:

**Lemma 4.6.** Let $V, W \subset T\Sigma$ be smooth distributions, satisfying $\omega(V, W) = 0$.

(a) The following are equivalent:

  (1) $V^\perp = W + V \cap K$;
  (2) $W^\perp = V + W \cap K$;
  (3) $V + (V^\perp)^\omega = V + W^\omega$;
  (4) $W + (W^\perp)^{-\omega} = W + V^{-\omega}$;
  (5) $\dim(\Sigma) + \dim(V \cap W \cap K) = \dim(V) + \dim(W)$.

(b) The following are equivalent:

  (1) $V^\perp = W \oplus V \cap K$;

(2) $W^{\perp} = V \oplus W \cap K$;

(3) $V + (V^{\perp})^{\omega} = V \oplus W^{\omega}$;

(4) $W + (W^{\perp})^{-\omega} = W \oplus V^{-\omega}$;

(5) $\dim(\Sigma) = \dim(V) + \dim(W)$;

and if these conditions are satisfied, then $V \cap W \cap K = 0$.

□

**Proof**. We will prove (a); part (b) is proven similarly. First note that:

$$\dim(V^{\perp}) = \dim(\Sigma) - \dim(V) + \dim(V \cap K),$$

$$\dim(W + V \cap K) = \dim(W) + \dim(V \cap K) - \dim(V \cap W \cap K).$$

Clearly, $W + V \cap K \subset V^{\perp}$; thus the two spaces are equal if, and only if, they have the same dimension. This shows that (1)⇔(5); and by symmetry (2)⇔(5).

On the other hand, we have that $\dim(V + (V^{\perp})^{\omega}) = \dim(\Sigma)$, and that:

$$\dim(V + W^{\omega}) = \dim(V) + \dim(W) - \dim(V \cap W \cap K);$$

therefore (3)⇔(5); and similarly, (4)⇔(5). ■

Next, we explain the geometric consequence of the condition in Lemma 4.6(b):

**Proposition 4.7** (Action criterion)**.** Let $\mu : (\Sigma, \omega) \to (M, L_M)$ be a forward Dirac map which is a surjective submersion. Assume that there exists an involutive subbundle $W \subset T\Sigma$, such that, in the notation of Corollary 4.1:

$$V^{\perp} = W \oplus V \cap K. \tag{5}$$

Then the Lie algebroid $L_M$ has an infinitesimal free action on $\mu : \Sigma \to M$ along the leaves of $W$; in other words, we have a Lie algebroid morphism covering $\mu$:

$$\begin{array}{ccc} W & \xrightarrow{a} & L_M \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\mu} & M \end{array}$$

which is a fibrewise isomorphism, and is given by:

$$a(v) = \mu_* v + \xi, \quad \text{where} \quad \mu^* \xi = \iota_v \omega.$$

□

**Proof**. That the map $a$ is indeed well-defined, and that it is a fibrewise isomorphism, follow from the direct sum decomposition Lemma 4.6 (b) (3):

$$\mu^* L_M = V \oplus W^{\omega}.$$

Finally, to show that $a$ is a Lie algebroid map, it suffices to show that:

$$a^{-1} \circ \mu^* : \Gamma(L_M) \longrightarrow \Gamma(W), \quad a^{-1} \circ \mu^*(u + \xi)_p := a^{-1}(u_{\mu(p)} + \xi_{\mu(p)})$$

is a Lie algebra homomorphism. For $u_1 + \xi_1,\ u_2 + \xi_2 \in \Gamma(L_M)$, let $v_1, v_2 \in \Gamma(W)$ be the unique elements so that $\mu_* v_i = u_i$ and $\mu^*\xi_i = \iota_{v_i}\omega$. In the terminology of Bursztyn [2013], this means that $u_i + \xi_i$ is $\mu$-related to $v_i + \iota_{v_i}\omega$; and therefore also their respective Dorfman brackets are $\mu$-related (see *loc.cit.*), i.e. if $[u_1 + \xi_1, u_2 + \xi_2] = u_3 + \xi_3 \in \Gamma(L_M)$, then $u_3 = \mu_*[v_1, v_2]$ and $\mu^*\xi_3 = \iota_{[v_1,v_2]}\omega$. Hence $a^{-1} \circ \mu^*$ is indeed a Lie algebra homomorphism. ■

**Remark 4.** In [Bursztyn et al., 2004, Definition 7.1], a forward Dirac submersion $\mu : (\Sigma, \omega) \to (M, L_M)$ is called a **presymplectic realization**, if

$$V \cap K = 0.$$

In this case, Proposition 4.3 implies that $W = V^{\perp}$ is automatically smooth and involutive, and the infinitesimal action of Proposition 4.7 recovers that of [Bursztyn et al., 2004, Corollary 7.3]. □

**Remark 5.** If $(M, L_M)$ is integrable by a (Hausdorff) presymplectic groupoid, then we are exactly in the situation of Proposition 4.7, where $\mu$ is the source map, and $W$ is the kernel of the target map. In this case, the action is the canonical action by left-invariant vector fields of the Lie algebroid $L_M$ on its Lie groupoid. The action is automatically **complete** (we will recall this notion in Corollary 4.8). Our main Theorem also fits in the setting of Proposition 4.7, and as explained in the Introduction, it is strongly related to this example. □

It was proven in [Crainic et al., 2004, Theorem 8], that the existence of a complete symplectic realizations of a Poisson manifold is equivalent to its integrability; but, as pointed out in [Crainic et al., 2004, Corollary 7], the proof depends only upon the existence of a complete action of the corresponding Lie algebroid. Applying these results in the same way as in the case of presymplectic realizations [Bursztyn et al., 2004, Remark 7.5], one obtains the following integrability result:

**Corollary 4.8** (Integrability criterion)**.** In the setting of Proposition 4.7, assume that the action $a : W \to L_M$ is complete, in the sense that: given $u + \xi \in \Gamma(L_M)$, where $u$ is a complete vector field, we have that $v := a^{-1}\mu^*(u + \xi) \in \Gamma(W)$ is also a complete vector field. Then $L_M$ is an integrable Dirac structure (by a groupoid which is not necessarily Hausdorff). Moreover, its source-simply connected Lie groupoid is Hausdorff if, and only if, the holonomy groupoid $\mathrm{Hol}(W) \rightrightarrows \Sigma$ of the foliation $W$ is Hausdorff. □

Next, the symmetry in the roles played by $V$ and $W$ implies the following:

**Proposition 4.9** (Dual pair criterion)**.** Let $(\Sigma, \omega)$ be a presymplectic manifold, and denote by $K := \ker \omega$. Consider two involutive subbundles $V, W \subset T\Sigma$ which satisfy the conditions of Lemma 4.6 (a). Assume that the corresponding foliations are simple, i.e. that their leaves are the fibres of two surjective submersions:

$$\mu_V : \Sigma \longrightarrow M_V \quad \text{and} \quad \mu_W : \Sigma \longrightarrow M_W.$$

Then there exist Dirac structures $L_V$ on $M_V$, and $L_W$ on $M_W$, which fit into the following diagram of forward Dirac maps:

$$(M_V, L_V) \xleftarrow{\mu_V} (\Sigma, \omega) \xrightarrow{\mu_W} (M_W, -L_W).$$

Moreover, the following relations hold:

$$\mu_V^* L_V = V + W^\omega \qquad \mu_W^* L_W = W + V^{-\omega} = (\mu_V^* L_V)^{-\omega}.$$

□

**Proof**. By applying Proposition 4.4 to $\mu_V$, $\omega$ and $W$, we obtain the Dirac structure $L_V$ on $M_V$ which satisfies $\mu_{V*} L_\omega = L_V$, and $\mu_V^* L_V = V + (V^\perp)^\omega = V + W^\omega$; similarly, by applying the same result to $\mu_W$, $-\omega$ and $V$, we obtain $L_W$ satisfying the corresponding conditions. ■

We believe that the following nomenclature should be used for the object constructed in Proposition 4.9:

**Definition 4.10.** Let $\omega$ be a closed 2-form on $\Sigma$, and consider a diagram of forward Dirac maps which are surjective submersions:

$$(M_1, L_1) \xleftarrow{\mu_1} (\Sigma, \omega) \xrightarrow{\mu_2} (M_2, -L_2). \tag{6}$$

We say that the diagram is a **full dual pair of Dirac structures**, if $V := \ker \mu_{1*}$, $W := \ker \mu_{2*}$ and $K := \ker \omega$ satisfy the conditions of Lemma 4.6 (a); and we call the full dual pair **robust**, if $V$, $W$ and $K$ satisfy the conditions of Lemma 4.6 (b). □

For example, in a presymplectic groupoid $(\Sigma, \omega) \rightrightarrows (M, L_M)$, the source and target maps give rise to a robust dual pair:

$$(M_1, L_M) \xleftarrow{\mathbf{s}} (\Sigma, \omega) \xrightarrow{\mathbf{t}} (M, -L_M).$$

**Remark 6.** In [Bursztyn et al., 2003, Definition 3.1], a **full pre-dual pair of Dirac structures** is defined as a diagram (6), where $\mu_1$ and $\mu_2$ are surjective, forward Dirac submersions, which satisfy the (equivalent) conditions:

$$V^\perp = W + K \quad \text{and} \quad W^\perp = V + K.$$

A full dual pair in the sense of Definition 4.10 is in particular a full pre-dual pair. Even though the notion of dual pairs is more restrictive, we believe that, in light of the discussion of this section, it is the one which is more natural to consider. □

**Example 4.11.** As a simple example of a full pre-dual pair which is not a full dual pair, consider

$$(\mathbb{R}^2, L_1) \xleftarrow{\mu_1} (\mathbb{R}^3, \omega) \xrightarrow{\mu_2} (\mathbb{R}^2, L_2),$$

where $\mu_1(x,y,z) = (x,y)$, $\mu_2(x,y,z) := (y,z)$, $\omega := \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z$, and $L_1$ and $L_2$ are the Dirac structures corresponding to the foliations $\langle \frac{\partial}{\partial x} \rangle$ and $\langle \frac{\partial}{\partial z} \rangle$ respectively. In this case, we have $V = \langle \frac{\partial}{\partial z} \rangle$, $V^\perp = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \rangle$, $W = \langle \frac{\partial}{\partial x} \rangle$, $K = \langle \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \rangle$, so:

$$V^\perp = W + K \neq W + K \cap V.$$

□

We note that, in our main Theorem, we have constructed a robust full dual pair:

**Corollary 4.12.** Any Dirac manifold $(M, L)$ is the leg of a robust full dual pair; more precisely, diagram (C) from the main Theorem is a robust dual pair. □

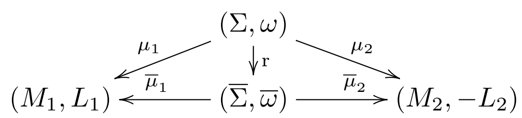
**Proof**. We have seen already that $\mathbf{s}$ and $\mathbf{t}$ are forward Dirac maps. Denote $V := \ker \mathbf{s}_*$ and $W := \ker \mathbf{t}_*$. Item (B) implies that for all $v \in \ker \mathbf{s}_*$ and $w \in \ker \mathbf{t}_*$, by (A), we have that the elements $v - \iota_v \omega$ and $w$ belong to the Dirac structure $\mathbf{t}^* L$, therefore, $\omega(v, w) = 0$; hence $\omega(V, W) = 0$. Finally, condition (5) of Lemma 4.6 (b) is obviously satisfied. ■

Our next observation is that, at least locally, any full dual pair of Dirac structures can be reduced to a robust one:

**Proposition 4.13** (Reduction Criterion)**.** Consider a full dual pair of Dirac manifolds:

$$(M_1, L_1) \xleftarrow{\mu_1} (\Sigma, \omega) \xrightarrow{\mu_2} (M_2, -L_2).$$

Then $V \cap W \cap K \subset T\Sigma$ is a smooth, involutive distribution. Assume that $V \cap W \cap K$ defines a simple foliation, i.e. that there is a surjective submersion $\mathrm{r} : \Sigma \to \overline{\Sigma}$ whose fibres are the leaves of $V \cap W \cap K$. Then the full dual pair above can be reduced to a *robust* full dual pair; namely, we have the following commutative diagram of surjective, forward Dirac submersions:



where the bottom line is a robust full dual pair, and $\overline{\omega}$ is a closed two-form on $\overline{\Sigma}$, such that $\omega = \mathrm{r}^*(\overline{\omega})$. □

**Proof**. First note that, while $K$ need not be smooth, it still satisfies the involutivity condition from Proposition 4.3. By Lemma 4.6 (a) (5), we have that $V \cap W \cap K$ has constant rank; the fact that it is indeed smooth follows because it can be represented as the kernel of the smooth vector bundle map:

$$W \longrightarrow \frac{V + (V^{\perp})^{\omega}}{V}, \quad u \mapsto [u + \iota_u \omega].$$

Moreover, $V \cap W \cap K$ is involutive because so are $V$, $W$ and $K$.

Assume now that there is a surjective submersion $\mathrm{r} : \Sigma \to M$ whose fibres are the leaves of $V \cap W \cap K$. By applying Proposition 3.3 to the three Dirac structures corresponding to the foliations $V$ and $W$ and to the closed two-form $\omega$, we deduce that there are foliations $\overline{V}$ and $\overline{W}$, and a closed two-form $\overline{\omega}$ on $\overline{\Sigma}$, such that $\mathrm{r}_* V = \overline{V}$ and $V = \mathrm{r}_*^{-1}\overline{V}$, $\mathrm{r}_* W = \overline{W}$ and $W = \mathrm{r}_*^{-1}\overline{W}$, and $\mathrm{r}_* L_\omega = L_{\overline{\omega}}$ and $\omega = \mathrm{r}^*\overline{\omega}$. Since $\ker \mathrm{r}_* \subset V \cap W$, and the fibres of r are connected, it follows that $\mu_1$ and $\mu_2$ are constant along the fibres of r; hence they factor as in the diagram $\mu_i = \overline{\mu}_i \circ \mathrm{r}$. Since the maps r, $\mu_1$ and $\mu_2$ are surjective submersions, this equality implies that the maps $\overline{\mu}_1$ and $\overline{\mu}_2$ are indeed smooth, and that they are surjective submersions. Next, note that these maps are indeed forward Dirac,

$$\overline{\mu}_{1*} L_{\overline{\omega}} = \overline{\mu}_{1*} \mathrm{r}_* L_\omega = \mu_{1*} L_\omega = L_1;$$

and similarly for $\overline{\mu}_2$. Clearly, $\overline{W} \subset \overline{V}^{\perp}$, and the robustness condition in the form of Lemma 4.6 (b) (5) is easily verified. ■

Next, we note the following relation between the presymplectic realization condition, the robustness condition, and the property that one of the legs in a dual pair is a Poisson structure:

**Proposition 4.14.** Consider a full dual pair of Dirac structures:

$$(M_1, L_1) \xleftarrow{\mu_1} (\Sigma, \omega) \xrightarrow{\mu_2} (M_2, -L_2).$$

Then:

(a) $L_2$ is a Poisson structure if, and only if, $W = V^{\perp}$.

(b) $\mu_1$ is a presymplectic realization if, and only if, the dual pair is robust and $L_2$ is a Poisson structure.

□

**Proof**. (a) By Corollary 4.1, the Dirac structure $L_2$ is Poisson if and only if $K \subset W$; and since $V^{\perp} = W + V \cap K$, this is equivalent to $W = V^{\perp}$.

(b) Since $V^\perp = W + V \cap K$, the following holds trivially:

$$V \cap K = 0 \iff \big(V^\perp = W \oplus V \cap K \ \text{ and } \ V^\perp = W\big),$$

i.e. $\mu_1$ is a presymplectic realization if and only if the dual pair is robust and $L_2$ is a Poisson structure. ■

Proposition 4.13 gives a procedure for reducing to dual pairs in which one component is a presymplectic realization:

**Corollary 4.15** (Reduction to presymplectic realizations)**.** Let $\mu_1 : (\Sigma, \omega) \to (M, L)$ be a forward Dirac map which is a surjective submersion. Assume that $V^\perp$ is a smooth distribution. Then $V \cap K$ is also a smooth distribution, and both $V^\perp$ and $V \cap K$ are involutive.

(a) If the foliation corresponding to $V^\perp$ is simple, i.e. if there exists a surjective submersion $\mu_2 : \Sigma \to N$ whose fibres are the leaves of $V^\perp$, then there exists a Poisson structure $\pi_N$ on $N$ which fits into the full dual pair

$$(M, L) \xleftarrow{\mu_1} (\Sigma, \omega) \xrightarrow{\mu_2} (N, -\pi_N).$$

(b) If, in addition to (a), $V \cap K$ is also a simple foliation, i.e. if its leaves are the fibres of a surjective submersion $\mathrm{r} : \Sigma \to \overline{\Sigma}$, then we can reduce to a robust full dual pair, which fits into the commutative diagram of forward Dirac submersions:

$$\begin{array}{ccccc} & \overset{\mu_1}{\swarrow} & (\Sigma, \omega) & \overset{\mu_2}{\searrow} & \\ & & \downarrow \mathrm{r} & & \\ (M, L) & \xleftarrow{\overline{\mu}_1} & (\overline{\Sigma}, \overline{\omega}) & \xrightarrow{\overline{\mu}_2} & (N, -\pi_N) \end{array}$$

where the map $\mu_1$ is a presymplectic realization.

□

**Example 4.16.** Consider the forward Dirac map $\mu_1 : (\mathbb{R}^3, \omega) \to (\mathbb{R}^2, L_1)$ from Example 4.11. Then $\mu_1$ is a presymplectic realization, i.e. $V \cap K = 0$, and $V^\perp$ is the simple foliation corresponding to the fibers of $\mu_3 : \mathbb{R}^3 \to \mathbb{R}$, $\mu_3(x, y, z) := y$. By Corollary 4.15, we obtain the robust dual pair (compare with the pre-dual pair in Example 4.11):

$$(\mathbb{R}^2, L_1) \xleftarrow{\mu_1} (\mathbb{R}^3, \omega) \xrightarrow{\mu_3} (\mathbb{R}, T^*\mathbb{R}).$$

□

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## References

Bursztyn, H. A brief introduction to Dirac manifolds. *Geometric and topological methods for quantum field theory*, Cambridge Univ. Press, Cambridge (2013): 4–38.

Bursztyn, H., Crainic, M., Weinstein, A. and Zhu, C. Integration of twisted Dirac brackets. *Duke Math. J.* Volume 123, Number 3 (2004): 549–607.

Bursztyn, H., Radko, O. Gauge equivalence of Dirac structures and symplectic groupoids. *Annales de l'Institut Fourier* Volume 53, Issue 1 (2003): 309–337.

Crainic, M. and Fernandes, R. L. Integrability of Lie brackets. *Annals of Mathematics* 157 (2003): 575–620.

Crainic, M. and Fernandes, R. L. Integrability of Poisson brackets. *J. Diff. Geom.* 66 (2004): 71–137.

Crainic, M. and Mărcuț, I. On the existence of symplectic realizations. *J. Symplectic Geom.* 9, no. 4 (2011): 435–444.

Coste, A., Dazord, P., Weinstein, A. Groupoïdes symplectiques. *Publ. Dép. Math. Nouvelle Ser. A* 2 (1987): 1–62.

Frejlich, P. and Mărcuț, I. The Normal Form Theorem around Poisson Transversals. preprint, `http://arxiv.org/abs/1306.6055`

Frejlich, P. and Mărcuț, I. Normal forms for Poisson maps and symplectic groupoids around Poisson transversals. preprint, `http://arxiv.org/abs/1508.05670`

Gualtieri, M. Generalized complex geometry. *Ann. of Math.* (2), 174 no. 1 (2011): 75–123.

Libermann, P. Problèmes d'équivalence et géométrie symplectique. *Astérisque* 107-108 (1983): 43–68.

Stefan, P. Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc.* (3) 29 (1974): 699–713.

Wade, A. Poisson fiber bundles and coupling Dirac structures. *Ann. Global Anal. Geom.* 33, no. 3 (2008): 207–217.

Weinstein, A. The local structure of Poisson manifolds. *J. Diff. Geom.* 18 (1983): 523–557.