EXOTIC OPEN 4-MANIFOLDS WHICH ARE NON-LEAVES

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Abstract. We study the possibility of realizing exotic smooth structures on punctured simply connected 4-manifolds as leaves of a codimension 1 foliation on a compact manifold. In particular, we show the existence of a continuum of smooth open 4-manifolds which are not diffeomorphic to any leaf of a codimension 1 transversely analytic foliation on a compact manifold. These examples include some exotic $\mathbb{R}^4$'s and exotic cylinders $S^3 \times \mathbb{R}$.

Introduction

The stunning results of Donaldson [10] and Freedman [11] provided the existence of exotic smooth structures on $\mathbb{R}^4$, which is known to be the unique euclidean space with this property. This is in fact also true [3] for an open 4-manifold with a collarable end. The fact that these structures can arise in 4-dimensional manifolds has implications for physics (see e.g. [1, 22]), i.e., what if our space-time carries an exotic structure? Since the exotic family was discovered in the 1980s, nobody has been able to find an explicit and useful exotic atlas. It is worthy of interest to obtain alternative explicit descriptions of these exotica.

An open manifold which is realizable as a leaf of a foliation in a compact manifold must satisfy some restrictions. Since the ambient is compact, an open manifold has to accumulate somewhere, and this induces recurrences and "some periodicity" on its ends.

Before reviewing the history of realizability of open manifolds as leaves, we now state our main results. Let $\mathcal{Z}$ be the set of open topological 4-manifolds (up to homeomorphism) obtained by removing a finite non-zero number of points from a closed, connected, simply connected 4-manifold. In Section 3 we shall define a class $\mathcal{Y}$ of smooth manifolds (up to diffeomorphism), each with at least one exotic end, whose underlying topological manifolds belong to $\mathcal{Z}$. To give some idea of $\mathcal{Y}$, we remark that every manifold in $\mathcal{Z}$ that has at least two ends, or is obtained by removing a single point from a smooth closed manifold, is the underlying topological manifold of an uncountable family of diffeomorphically distinct manifolds in $\mathcal{Y}$.

Theorem 1. *If $Y \in \mathcal{Y}$ is a leaf in a $C^{1,0}$ codimension one foliation of a closed 5-manifold, then it is a proper leaf and each connected component of the union of the leaves diffeomorphic to Y fibers over the circle with the leaves as fibers.*

Theorem 2. *For any manifold $Y \in \mathcal{Y}$ there exists an uncountable subset $\mathcal{Y}_Y \subset \mathcal{Y}$ of manifolds homeomorphic to Y that are not diffeomorphic to any leaf of a transversely analytic codimension 1 foliation of a compact manifold.*

The following result of independent interest, which uses the theory of levels and depth described in Section 3, will be used in the proof of Theorem 2.

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Theorem 3. *Every open manifold has at most countably many smooth structures (up to diffeomorphism) that can be leaves of finite depth in a codimension one transversely C^{1+Lip} foliation of a compact manifold.*

Remark that in this theorem, there is no restriction in the dimension of the foliation; it can be greater than 4.

Next we review some of the history of leaves and non-leaves. It was shown by J. Cantwell and L. Conlon [7] that every open surface is homeomorphic (in fact, diffeomorphic) to a leaf of a foliation on each closed 3-manifold. The first examples of topological non-leaves were due to E. Ghys [15] and T. Inaba, T. Nishimori, M. Takamura, N. Tsuchiya [20]; these are highly topologically non-periodic open 3-manifolds which cannot be homeomorphic to leaves in a codimension 1 foliation in a compact manifold. Years later, O. Attie and S. Hurder [2], in a deep analysis of the question, found simply connected examples of non-leaves, non-leaves which are homotopy equivalent to leaves and even a Riemannian manifold which is not quasi-isometric to a leaf in arbitrary codimension. This final example follows the line of the work of A. Phillips and D. Sullivan [24] and T. Januszkiewicz [21]. We remark that these later examples are 6-dimensional.

C.L. Taubes [27] showed that the smooth structure of some of the exotic $\mathbb{R}^4$'s is, in some sense, non-periodic at infinity, and this leads to the existence of uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$. It is an open problem whether an exotic $\mathbb{R}^4$—and, by extension, any given open manifold with a similar exotic end smooth structure—can be diffeomorphic to a leaf of a foliation on a compact manifold. By a simple cardinality argument, most exotic $\mathbb{R}^4$'s cannot be covering spaces of closed smooth 4-manifolds by smooth covering maps since the diffeomorphism classes of smooth closed manifolds are at most countable. A direct proof can be given for some classes of exotic $\mathbb{R}^4$'s [28]. All these results motivated a folklore conjecture in foliation theory suggesting that these exotic structures cannot occur in leaves of a foliation in a compact manifold.

The main difference between some exotic $\mathbb{R}^4$'s (called *large*) and the standard $\mathbb{R}^4$ is the fact that they cannot embed smoothly in a standard $\mathbb{R}^4$. An important property for a large exotic $\mathbb{R}^4$ is to describe what are the simplest spin manifolds (in the sense of the second Betti number) in which it can be embedded; this is measured by the invariant defined by L. Taylor [28], which provided the first direct tool to show that some exotic $\mathbb{R}^4$'s cannot be non-trivial covering spaces. We can subdivide large exotic $\mathbb{R}^4$ in two classes: those that can be embedded in spin closed 4-manifolds with hyperbolic intersection form and those which cannot; our arguments differ slightly between these two families.

These particular exotica have a good control on the end structure and we can use them to perturb the standard end of a punctured smooth 4-manifold. We adapt Ghys' procedure in [15] to show some necessary conditions for such structures to be leaves of a codimension 1 foliation on a compact manifold. Precisely, Theorem 1 shows that if one of these manifolds is a leaf, it must be a proper leaf, and the union of leaves diffeomorphic to it is a saturated open set such that each of its connected components fibers over the circle. In Theorem 2, we complete this analysis in the case of analytic foliations (those where the transverse coordinate changes are analytic maps), by using the theory of levels [5], which gives some control of the end-recurrences of the leaves, and the result due to G. Hector showing that, in the case of an exceptional local minimal set, the stabilizer group of any gap is cyclic.

The paper is organized as follows:

- The first section is devoted to exotic structures on open 4-manifolds, particularly in $\mathbb{R}^4$. This is in fact a brief exposition of the results in [4, 14, 16, 28]. Here we define the particular exotic structures considered on $\mathbb{R}^4$ and show some of their properties.
- In the second section we prove Theorem 1 and give necessary conditions for some exotic punctured simply connected 4-manifolds to be diffeomorphic to leaves, following Ghys' method of proof [15].
- In the third section we describe the theory of levels and use it to prove Theorem 3 and Theorem 3.1, showing the existence of an uncountable family of exotic manifolds which cannot be leaves at finite depth.
- In the fourth section we consider the analytic case and prove Theorem 2 which shows the existence of an uncountable family of exotic structures which are non-leaves.
- The last section includes some last remarks and open questions.

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1. Exotic structures on $\mathbb{R}^4$

In this section we construct a continuum of exotic structures in $\mathbb{R}^4$ which are non-periodic by Taubes' work. Later we shall need a better control of this structure, which is provided by the invariant defined by L. Taylor [28]. This introduction begins with a brief reminder of some known facts in 4-dimensional differential topology.

Theorem 1.1 (Freedman [11])**.** *Two simply connected closed* 4*-manifolds are homeomorphic if and only if their intersection forms are isomorphic and have the same modulo* 2 *Kirby-Siebemann invariant. In particular, simply connected smooth closed* 4*-manifolds are homeomorphic if and only if their intersection forms are isomorphic.*

Theorem 1.2 (Donaldson [10])**.** *If a smooth closed simply connected* 4*-manifold has a definite intersection form then it is equivalent to a diagonal form.*

Definite symmetric bilinear unimodular forms are not classified and it is known that the number of equivalence classes grows at least exponentially with the range. Indefinite unimodular forms are classified [26]: two indefinite forms are isomorphic if they have the same range, signature, and parity. There are canonical representatives for the indefinite forms; in the odd case the form is diagonal and in the even case $H_2(M, \mathbb{Z})$ splits into invariant subspaces where the intersection form is either E_8 or H. These canonical representatives are denoted as usual with the notation $\oplus m[+1] \oplus n[-1]$ for the odd case and $\oplus \pm mE_8 \oplus nH$ for the even one.

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} ; \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For each symmetric bilinear unimodular form there exists at least one topological simply connected closed 4-manifold with an isomorphic intersection form. But this is no longer true for the smooth case, as Donaldson's theorem asserts. It is an open problem what unimodular forms can be realized in smooth simply connected closed 4-manifolds. It is known that for a smooth simply connected even 4-manifold the number of "E_8 blocks" must be even (Rokhlin's theorem). It is possible to say more, as in Furuta's theorem [13] which will be useful in this section.

Theorem 1.3 (Furuta [13])**.** *If M is a smooth simply connected closed 4-manifold with an intersection form equivalent to $\oplus \pm 2mE_8 \oplus nH$ and $m \neq 0$, then $n \geq 2m+1$.*

Another important tool for this section is the "end sum" construction. For open manifolds this is analogous to the connected sum of closed manifolds. Given two open smooth manifolds M and N we choose two smooth proper paths $c_1 : [0,\infty) \to M$ and $c_2 : [0,\infty) \to N$, each of them defining one end in M and N respectively. Let V_1 and V_2 be tubular neighborhoods of $c_1([0,\infty))$ and $c_2([0,\infty))$. The boundaries of these neighborhoods are clearly diffeomorphic to $\mathbb{R}^3$ and we can do a smooth sum by identifying these boundaries so as to respect the orientations. This will be called the end sum of M and N associated to c_1 and c_2, and it is denoted by $M \natural N = M \setminus V_1 \bigcup_\partial N \setminus V_2$. In the case where N and M are both homeomorphic to $\mathbb{R}^4$, the smooth structure of $M \natural N$ only depends in the isotopy class of the paths c_1 and c_2. End sum was the first technique which made it possible to find infinitely many exotic structures on $\mathbb{R}^4$ [16] and it is an important tool for dealing with the problem of generating infinitely many smooth structures on open 4-manifolds [3, 14].

Let us recall an important theorem of M.H. Freedman which is the main tool to determine when a manifold is homeomorphic to $\mathbb{R}^4$.

Theorem 1.4 (Freedman [11])**.** *An open 4-manifold is homeomorphic to $\mathbb{R}^4$ if and only if it is contractible and simply connected at infinity.*

Now we describe the construction of an exotic $\mathbb{R}^4$ whose end structure is diffeomorphic to the end of a punctured $-E_8 \oplus -E_8$ manifold, as in Taubes' work [27].

Let M_0 be the $K3$ Kummer surface. It is known that the intersection form of M_0 can be written as $-2E_8 \oplus 3H$, where the six elements in $H_2(M_0,\mathbb{Z})$ spanning the summand $3H$ can be represented by six Casson handles C_i attached to a 4-dimensional ball B^4 inside M_0. Let $U = \operatorname{int}(B^4 \cup \bigcup_{i=1}^6 C_i)$ which is clearly homeomorphic to a punctured $\#^3 S^2 \times S^2$ by Freedman's theorem 1.1. Let S be the union of the cores of the Casson handles, which we consider to be inside $\#^3 S^2 \times S^2$. By theorem 1.4 the manifold $\mathbf{R} = \#^3 S^2 \times S^2 \setminus S$ is homeomorphic to $\mathbb{R}^4$. If this

R were standard then we could smoothly replace the $3H$ part in the intersection form of M_0 by a standard ball, so the resulting smooth closed manifold would have intersection form $-2E_8$, in contradiction to Donaldson's theorem 1.2, since $-2E_8$ is not isomorphic to a diagonal form. Let **K** be the compact set in **R** which is the bounded component determined by the boundary of a small neighborhood of S.

Notation 1.5. Let $\psi : \mathbb{R}^4 \to \mathbf{R}$ be a homeomorphism. Let us denote $\mathbf{K}_t = \psi(D(0,t))$, where $D(0,t)$ is the standard closed disk of radius t, and consider the smooth structure in $\mathring{\mathbf{K}}_t$ induced by **R**.

We now present a version of Taubes' theorem which suffices for our purposes.

Definition 1.6 (Periodic end)**.** Let M be an open smooth manifold with one end homeomorphic to $S^3 \times (0,\infty)$. We say that this end is *smoothly periodic* if there exists an unbounded domain $V \subset M$ homeomorphic to $S^3 \times (0,\infty)$ and a diffeomorphism $h : V \to V$ such that $h^n(V)$ defines the given end (i.e., $\{h^n(V)\}$ is a neighborhood base for the end).

Note that this notion of smoothly periodic end is a particular case of admissible periodic ends considered in [27].

Theorem 1.7 (Taubes [27])**.** *Let M be an open smooth simply connected 4-manifold with definite intersection form and exactly one end. If the end of M is homeomorphic to $S^3 \times (0,\infty)$ and smoothly periodic, then the intersection form is isomorphic to a diagonal form. As a consequence there exists $r_0 > 0$ such that, for any $t, s > r_0$, $t \neq s$, $\mathbf{K}_t$ is not diffeomorphic to $\mathbf{K}_s$.*

Definition 1.8 (Taylor [28])**.** Let E be an exotic $\mathbb{R}^4$. Let $Sp(E)$ be the set of closed smooth Spin 4-manifolds N with trivial or hyperbolic intersection form in which E embeds smoothly. Define $b_E = \infty$ if $Sp(E) = \emptyset$, or else:

$$2b_E = \min_{N \in Sp(E)} \{\beta_2(N)\} .$$

Let $\mathcal{E}(E)$ be the set of topological embeddings $e : D^4 \to E$ such that e is smooth in the neighborhood of some point of the boundary and $e(\partial D^4)$ is (topologically) bicollared. Set $b_e = b_{e(\mathring{D}^4)}$ where $e(\mathring{D}^4)$ has the smooth structure induced by E. The *Taylor-index* of E is defined to be

$$\gamma(E) = \max_{e \in \mathcal{E}(E)} \{b_e\} .$$

For a spin manifold M, the Taylor-index of M is the supremum of the Taylor-indices of all the exotic $\mathbb{R}^4$'s embedded in M.

Proposition 1.9 (Proposition 2.2 [28])**.** *Let $e_i \in \mathcal{E}(E)$, $i = 1, \ldots, k$, be pairwise disjoint topological disks in E. Then there exists $e \in \mathcal{E}(E)$ such that $e(\mathring{D}^4)$ is diffeomorphic to $\natural_{i=1}^k e_i(\mathring{D}^4)$.*

Proposition 1.10 (Example 5.6 [28])**.** *We can assume $\mathbf{K} \in \mathcal{E}(\mathbf{R})$ (a topologically embedded ball with the required properties). Then $b_{\mathbf{K}} = \gamma(\mathbf{R}) = 3$ and, for any $e \in \mathcal{E}(\mathbf{R})$ such that $e(D^4) \cap \mathbf{K} = \emptyset$, $e(\mathring{D}^4)$ is not diffeomorphic to $\mathbf{K}$.*

Corollary 1.11 (Taylor)**.** **R** *cannot be a non-trivial covering space of any smooth manifold.*

Proof. Otherwise there would exist a diffeomorphism acting properly on $\mathbf{R}$, so some power of this covering map would disconnect $\mathbf{K}$ from a diffeomorphic copy of itself, contradicting the above proposition. □

Proposition 1.12 (Theorem 5.3 [28])**.** *The Taylor-index of* $\natural^n\mathbf{K}$ *tends to* ∞*. In particular, the existence of infinitely many pairwise disjoint sets diffeomorphic to* $\mathbf{K}$ *implies that the manifold cannot be embedded in a compact spin manifold with hyperbolic intersection form. For instance,* $\mathbf{R}_\infty = \natural_{i=1}^\infty \mathbf{R}$*.*

Remark 1.13. In Example 5.10 in [28], uncountably many non-diffeomorphic smooth structures $\mathbb{R}^4$ with infinite Taylor index are exhibited. Each element of this family is end diffeomorphic to $\mathring{\mathbf{K}}_t\natural\mathbf{R}_\infty$ for all sufficiently large t.

On the other hand, $\mathbf{R}_\infty$ can be a non-trivial covering space of an open manifold. In fact $\mathbf{R}_\infty$ admits several free actions (see e.g. [16, 17]); for example, $\mathbf{R}_\infty$ is diffeomorphic to the end sum $\natural_{i\in\mathbb{Z}}\mathbf{R}$, which admits an obvious free action of $\mathbb{Z}$ whose quotient is $\mathbf{R}$.

Another way to describe the pathologies of large exotic $\mathbb{R}^4$'s is by considering smooth 3-submanifolds disconnecting large compact sets from the end. The discussion of the Taylor index above shows that these disconnecting manifolds cannot be smooth spheres, so invariants associated to these submanifolds give a measure of the complexity of the exotic structure. This is exactly what is measured by the engulfing index defined by Ž. Bižaca and R. Gompf [4, 14].

Notation 1.14. Let X be an open manifold with an isolated end e and let $K \subset X$ be a compact set. Let $\Sigma|_{eK}$ denote a smooth embedded 3-submanifold Σ of X which disconnects K from e. This means that $X \setminus \Sigma$ has two connected components, one of which contains K and the other of which is a neighborhood of e. It is clear from basic differential topology theory that for a given K such a Σ does exist. If X has exactly one end e then we denote $\Sigma|_{eK}$ by $K \subset X$.

Remark 1.15. Recall that C^1 submanifolds are isotopic to smooth (C^∞) submanifolds arbitrarily close to them, so in the definition of the complexity it is not necessary to consider whether the separating submanifold Σ is C^1 or smooth.

Definition 1.16 (Engulfing index)**.** Let X be a smooth manifold and let e be an isolated end of X. The *engulfing index* of X *in the direction of* e, denoted by $c_e(X)$, is the number (possibly ∞) given by the following expression:

$$c_e(X) = \sup_{K\subset X}\{\inf_{\Sigma|_e K} b_1(\Sigma)\}\ ,$$

where K runs over the compact sets in X, Σ runs over the embedded smooth closed 3-submanifolds disconnecting K from the end e, and $b_1(\Sigma)$ is the first Betti number of Σ. When the end being considered is clear from the context (for instance when there is only one end or only one is not standard) we shall use the notation $c(X)$.

Proposition 1.17. [4, 14] $c(\natural_1^n\mathbf{R}) > 2n$ *and* $c(\mathbf{R}_\infty) = \infty$*.*

For the sake of completeness we shall sketch the proof of this proposition. The proof splits into two parts. First of all the existence of an exotic $\mathbb{R}^4$ with positive complexity must be shown. Then it is shown that the end sum of these particular exotica produces exotic $\mathbb{R}^4$'s with higher complexity. Thus an infinite end sum will produce an exotic $\mathbb{R}^4$ with infinite complexity.

We show that $\mathbf{R}$ has complexity greater than 2. We want to show that any smooth 3-submanifold Σ separating $\mathbf{K}$ from the end has first Betti number $\beta_1(\Sigma) > 2$. Assume that $\beta_1(\Sigma) \leq 2$.

Let N be the compact 4-manifold bounded by Σ inside $\mathbf{R}$. In the $K3$ surface M_0 we can obtain a smooth copy of Σ separating the $3H$ component represented by S from the $-2E_8$ component, and we let M be the 4-manifold corresponding to $2E_8$ bounded by Σ in M_0. Then we can identify the boundaries and obtain a smooth closed manifold $Y = M \cup_\Sigma N$, which must be spin since all the factors considered have even intersection forms. Let us consider the Mayer-Vietoris sequence associated to M and N with rational coefficients:

$$\cdots \to H_2(\Sigma) \xrightarrow{\ \varphi\ } H_2(M) \oplus H_2(N) \xrightarrow{\ \psi\ } H_2(Y) \to H_1(\Sigma) \to \cdots$$

By Poincare duality $H_2(\Sigma) \approx H_1(\Sigma)$ and they have at most two generators. The key observation is the fact that $H_2(M, \Sigma) = -2E_8$ (understanding this notation as the corresponding subspace of $H_2(M)$ invariant by $-2E_8$) and $H_2(N, \Sigma) = 0$. From the exact homology sequence of the pair (M, Σ)

$$\cdots \to H_2(\Sigma) \xrightarrow{\ i_*\ } H_2(M) \xrightarrow{\ j_*\ } H_2(M, \Sigma) \xrightarrow{\ \partial\ } H_1(\Sigma) \to \cdots$$

we see that the homology 2-classes in $H_2(M)$ that become zero in $H_2(M, \Sigma)$ come from 2-classes of $H_2(\Sigma)$. A similar result holds for $H_2(N)$. In the Mayer-Vietoris sequence the image of ψ is generated by $H_2(M, \Sigma) = -2E_8$ and at most two elements in the image of φ, since $j_* \circ i_* H_2(\Sigma) = 0$. Thus $H_2(Y)$ consists of the classes in $-2E_8$, at most two other generators in the image of ψ, and at most two generators whose images in $H_1(\Sigma)$ are non-zero. Therefore the intersection form of Y is at most $-2E_8 \oplus 2H$, with only two copies of H, and this contradicts Furuta's theorem. Thus $\beta_1(\Sigma) > 2$, so the complexity of R is also greater than 2.

A similar argument applies to $\natural_{i=1}^n \mathbf{R}$ to show that $\beta_1(\Sigma) > 2n$. In this case we could construct a smooth closed spin manifold with intersection form at best $-2nE_8 \oplus 2nH$ (the non-optimal case would have less hyperbolics and more E_8's) which contradicts Furuta's theorem again. An inductive argument yields the result for $\mathbf{R}_\infty$.

Remark 1.18. Remark that the same uncountable family of non-diffeomorphic structures given in Example 5.10 in [28] have infinite engulfing index.

2. Exotic simply connected smooth 4-manifolds that are leaves

In this section we give necessary conditions for certain exotic simply connected smooth manifolds to be diffeomorphic to leaves of a codimension 1 foliation in a compact manifold. As mentioned in the introduction, we define $\mathcal{Z}$ to be the set of open topological 4-manifolds (up to homeomorphism) which are obtained by removing a finite non-zero number of points from a closed, connected, simply connected 4-manifold.

Definition 2.1. Let $\mathcal{Y}$ be the set of smooth manifolds Y (up to diffeomorphism) that are homeomorphic to members of $\mathcal{Z}$ such that

(1) Y has an end diffeomorphic to the end of a finite end sum $\natural_{i=1}^k \mathbf{R}$, to $\natural_{i=1}^k \mathring{\mathbf{K}}_t$ or to $\mathring{\mathbf{K}}_t \natural \mathbf{R}_\infty$ with $t > r_0$ (with the notations of the previous section), and
(2) if $H_2(Y) = 0$, then Y has only one exotic end and the other ends (if there are any) are standard.

(3) In the particular case where Y is homeomorphic to $\mathbb{R}^4$ we only consider smooth structures with finite Taylor-index.

Example 2.2. If Z_Y is a simply connected smooth closed 4-manifold that is not homeomorphic to S^4, then $Y = Z_Y \# \mathbf{R} \in \mathcal{Y}$, since Freedman's theorem 1.1 shows that $H_2(Y) \approx H_2(Z_Y) \neq 0$, and Y is homeomorphic but not diffeomorphic to $Z_Y \setminus \{*\}$ by Taubes' work. If $Z_Y = S^4$ then every non-trivial finite end sum of $\mathbf{R}$ belongs to $\mathcal{Y}$.

The goal of this section is the proof of Theorem 1, which states that if a leaf of a $C^{1,0}$ codimension one foliation of a compact manifold is diffeomorphic to $Y \in \mathcal{Y}$, then the leaf is proper and each connected component of the union of all leaves diffeomorphic to Y fibers over the circle with the leaves as fibers.

Remark 2.3. Regularity $C^{1,0}$ means that the leaves are tangent to a continuous hyperplane distribution of codimension 1.

Remark 2.4. It is well-known that every open 4-manifold obtained by removing a finite number of points from a closed manifold admits a differential structure with at most one exotic end. If $Y_0 \in \mathcal{Z}$ has at least two ends, then by forming end sums with $\natural_{i=1}^k \mathring{\mathbf{K}}_t$ or $\mathring{\mathbf{K}}_t \natural \mathbf{R}_\infty$ (with $t > r_0$), we obtain uncountably many non-diffeomorphic smooth manifolds $Y \in \mathcal{Y}$ that are homeomorphic to Y_0, all of them satisfying Theorem 1. The same holds for many manifolds $Y_0 \in \mathcal{Z}$ with one end.

In proving Theorem 1 we use the basic theory of codimension 1 foliations of smooth compact manifolds presented as integrable plane fields. Note that in this general situation there exists a smooth transverse one-dimensional foliation $\mathcal{N}$ and a biregular foliated atlas, i.e., one in which each coordinate neighborhood is foliated simultaneously as a product by $\mathcal{F}$ and $\mathcal{N}$. The transverse coordinate changes are only assumed to be continuous but the leaves can be taken to be smooth manifolds and the local projection along $\mathcal{N}$ of one plaque onto another plaque in the same chart is a diffeomorphism. Our basic tools are Dippolito's octopus decomposition and his semistability theorem [5, 9] as well as the trivialization lemma of G. Hector [19].

We assume that our foliation is transversely oriented, which is not a real restriction since all the manifolds considered are simply connected and therefore, by passing to the transversely oriented double cover, the foliation becomes transversely oriented. For a saturated open set U of $(M, \mathcal{F})$ we let $\hat{U}$ be the completion of U for the Riemannian metric of M restricted to U. The inclusion $i : U \to M$ clearly extends to an immersion $i : \hat{U} \to M$, which is at most 2-to-1 on the boundary leaves of $\hat{U}$. We shall use ∂^τ and $\partial^\pitchfork$ to denote the tangential and transverse boundaries, respectively.

Theorem 2.5 (Octopus decomposition [5, 9])**.** *Let U be a connected saturated open set of a codimension 1 transversely orientable foliation $\mathcal{F}$ in a compact manifold M. There exists a compact submanifold K (the nucleus) with boundary and corners such that*

(1) $\partial^\tau K \subset \partial^\tau \hat{U}$
(2) $\partial^\pitchfork K$ *is saturated for* $i^*\mathcal{N}$
(3) *the set $\hat{U} \setminus K$ is the union of finitely many non-compact connected components $B_1, \dots, B_m$ (the arms) with boundary, where each B_i is diffeomorphic*

to a product $S_i \times [0,1]$ by a diffeomorphism $\phi_i : S_i \times [0,1] \to B_i$ such that the leaves of $i^*\mathcal{N}$ exactly match the fibers $\phi_i(\{*\} \times [0,1])$.

(4) *the foliation $i^*\mathcal{F}$ in each B_i is defined by the suspension of a homomorphism from $\pi_1(S_i)$ to the group of homeomorphisms of $[0,1]$. Thus the holonomy in the arms of this decomposition is completely described by its action on a common complete transversal.*

Observe that this decomposition is far from being canonical, for the compact set K can be extended in many ways yielding other decompositions. We do not consider the transverse boundary of B_i to be a part of B_i; in particular, the leaves of $i^*\mathcal{F}_{|B_i}$ are open sets in leaves of $i^*\mathcal{F}$. Remark also that the word diffeomorphism will only be applied to open sets (of M or of leaves of $\mathcal{F}$); on the transverse boundaries the maps are only considered to be homeomorphisms.

Lemma 2.6 (Trivialization Lemma [19]). *Let J be an arc in a leaf of $\mathcal{N}$. Assume that each leaf meets J in at most one point. Then the saturation of J is diffeomorphic to $L \times J$, where L is a leaf of $\mathcal{F}$, and the diffeomorphism carries the bifoliation $\mathcal{F}$ and $\mathcal{N}$ to the product bifoliation of $L \times J$ (with leaves $L \times \{*\}$ and $\{*\} \times J$).*

Theorem 2.7 (Dippolito semistability theorem [5, 9]). *Let L be a semiproper leaf which is semistable on the proper side, i.e., there exists a sequence of fixed points for all the holonomy maps of L converging to L on the proper side. Then there exists a sequence of leaves L_n converging to L on the proper side and projecting diffeomorphically onto L via the fibration defined by $\mathcal{N}$.*

Let X be a neighborhood of the ends of $Y \in \mathcal{Y}$ identified with $\bigsqcup_{i=1}^n S^3 \times [0,\infty)$ such that the boundaries $\bigsqcup_{i=1}^n S^3 \times \{0\}$ are (topologically) bicollared in Y. Denote the connected components of X by X_n, where X_1 is an exotic end diffeomorphic to a finite end-sum of $\mathbf{R}$. Then we have the decomposition

$$Y = K_Y \cup X$$

where K_Y is the closure of $Y \setminus X$, so it is compact with boundary, and, in the case that Y is not homeomorphic to $\mathbb{R}^4$ with finite punctures, it has non-trivial second homology.

Now we have enough information to begin to follow the line of reasoning of Ghys [15]. For the rest of this section we assume that $Y \in \mathcal{Y}$ is diffeomorphic to a leaf, and we shall find some constraints.

Definition 2.8. We say that a leaf $L \in \mathcal{F}$ *contains a vanishing cycle* if there exists a connected 3-cycle $\Sigma \subset L$ that is non-null-homologous on L and a family of connected 3-cycles $\{\Sigma(n) \mid n \in \mathbb{N}\}$ on L that are null-homologous on L and converge to Σ along leaves of the transverse foliation $\mathcal{N}$.

Proposition 2.9. *In a (transversely oriented) C^0 codimension one foliation of a compact manifold, no simply connected leaf L with at least two ends contains an embedded vanishing cycle Σ homeomorphic to S^3 in an end homeomorphic to $S^3 \times (0,\infty)$.*

To prove this, we shall use a weak generalization of Novikov's theorem on the existence of Reeb components, Theorem 4 of [25]. Recall that a (generalized) Reeb component with connected boundary is a compact $(k+1)$-manifold with a codimension one foliation such that the boundary is a leaf and the interior fibers over the circle with the leaves as fibers. Suppose we are given a compact $(k+1)$-manifold M with a transversely oriented codimension one foliation $\mathcal{F}$ and a transverse foliation $\mathcal{N}$, and also a connected closed k-dimensional manifold B with a bifoliated map $h : B \times [a, b] \to M$, where $[a, b]$ is an interval in the real line, such that $h(B \times \{t\})$ is contained in a leaf L_t of $\mathcal{F}$ for every $t \in [a, b]$ and $h_a : B \to L_a$ is an embedding, where $h_t(x) = h(x, t)$.

Theorem 2.10 (See [25], Theorem 4)**.** *If for every $t \in (a, b]$, $B_t = h_t(B)$ bounds a compact connected region in L_t, but B_a does not bound on L_a, then L_a is the boundary of a Reeb component whose interior leaves are the leaves L_t for $t \in (a, b]$.*

Proof of Proposition 2.9. Let L be a simply connected leaf with at least two ends in a C^0 codimension one foliation of a compact manifold M, and suppose that L contains an (embedded) vanishing cycle Σ homeomorphic to S^3 in an end homeomorphic to $S^3 \times (0, \infty)$. Then there is a sequence $\Sigma(n)$ of null-homologous 3-cycles on L converging to the non-null-homologous 3-cycle Σ along a transverse foliation $\mathcal{N}$. Let $\Sigma \times [-1, 1]$ be identified with a bifoliated neighborhood of Σ so that Σ is identified with $\Sigma \times \{0\}$. We may assume without loss of generality that infinitely many of the cycles $\Sigma(n)$ are on the positive side of Σ and are contained in a fixed domain of the end. Each $\Sigma(n)$ bounds a compact region embedded in L. Let S_+ (resp., S_-) be the set of numbers $t \in (0, 1]$ such that in the leaf L_t that contains $\Sigma_t = \Sigma \times \{t\}$, Σ_t bounds a compact region C_t on the positive (resp., negative) side of Σ_t. Note that C_t must be 1-connected, so S_+ and S_- are open by Reeb stability.

Now there exists an $\epsilon > 0$ such that $S_+ \cap S_- \cap (0, \epsilon) = \emptyset$, for any leaf containing Σ_t with $t \in S_+ \cap S_- \cap (0, \epsilon)$ would be the union of two compact regions and therefore compact, so if no such ϵ existed, L would be a limit of compact leaves and therefore compact, which is false. Now at least one of S_+ and S_- has 0 as a limit point—say it is S_+. If there existed $c > 0$ such that $(0, c) \subset S_+$, then by Theorem 2.10, the leaf L would be the boundary of a (generalized) Reeb component, which is compact, again giving a contradiction. Hence there must exist open intervals $(a, b) \subset (0, 1)$ which are connected components of S_+ arbitrarily close to 0, so $(a, b) \subset S_+$ and $a \notin S_+$. For $a < \epsilon$, we also have $a \notin S_-$ (for otherwise $S_+ \cap S_- \cap (0, \epsilon)$ would not be empty), so Σ_a does not bound on L_a. Applying Theorem 2.10 again, we find that the leaf L_a must be the boundary of a Reeb component and therefore compact. Since this must hold for values of a arbitrarily close to 0, L must be compact, which is false, so L cannot contain a vanishing cycle with the required properties. □

Proposition 2.11. *Let $\mathcal{F}$ be a codimension one $C^{1,0}$ foliation in a compact 5-manifold M. If there exists a leaf L of $\mathcal{F}$ diffeomorphic to $Y \in \mathcal{Y}$, then L is a proper leaf without holonomy.*

Proof. Since L is simply connected, it is a leaf without holonomy. We also observe that L has a saturated neighborhood not meeting any compact leaves, since a limit leaf of compact leaves is compact. Consider first the case where Y is not homeomorphic to $\mathbb{R}^4$ with a finite number of punctures. We consider K_Y to be a subset of L, by using the diffeomorphism from Y to L. By Reeb stability there exists a neighborhood U of K_Y bifoliated diffeomorphically as a product. If L meets U in more than one connected component then there exists a compact subset $B \subset L$ homeomorphic to K_Y (via the transverse projection in U) and disjoint from K_Y. This is impossible since the inclusion $i_0 : K_Y \hookrightarrow L$ induces an isomorphism

$i_{0*} : H_2(K_Y) \to H_2(L)$, and the Mayer-Vietoris sequence shows that B would give an additional non-trivial summand in $H_2(L)$. So in this case L is a proper leaf.

In the case where Y is homeomorphic to $\mathbb{R}^4$ with k punctures we have to split the proof into two parts. Firstly, if the Taylor-index is finite then we can find a compact set K in Y diffeomorphic to $\natural_1^n \mathbf{K}_s$ (as in the last section, since the punctures can be taken to be far away from K). If L is not proper, the application of Reeb stability will produce infinitely many pairwise disjoint copies of $\mathbf{K}_s$ and therefore $\gamma(Y) = \infty$ by Proposition 1.12, contradicting the fact that Y is embedded in a smooth manifold homeomorphic to $3k\#S^2 \times S^2$.

On the other hand, assume that Y has an exotic end with infinite Taylor-index (and therefore infinite engulfing index as well) and that the other ends are standard. Let Q be a compact set in L homeomorphic but not diffeomorphic to $S^3 \times [0,1]$ in a domain of the exotic end that is homeomorphic to $S^3 \times (0,\infty)$ such that Q disconnects the exotic end from the standard ones. By Reeb stability there exists a neighborhood of Q bifoliated as a product $Q \times (-1,1)$ (with the original Q identified with $Q \times \{0\}$) where the projection of a tangential leaf to another in this neighborhood is a diffeomorphism. If $L \cap Q \times (-1,1)$ contains a non-trivial subsequence $Q \times \{t_n\}$ with t_n tending to 0, then these fibers belong to a domain of the exotic end and two situations may occur:

(1) for all sufficiently large values of n, $Q \times \{t_n\}$ does not disconnect the exotic end from the standard ones. Since for these values of n, $Q \times \{t_n\}$ bounds in S^4 (consider $S^3 \times (-1,1)$ to be embedded in S^4) and it does not separate the two ends, it also bounds in $S^3 \times (-1,1)$. Thus L would contain a vanishing cycle, which is not possible by Proposition 2.9.
(2) $Q \times \{t_n\}$ disconnects the ends of L for some subsequence of t_n. Let Σ be a smooth closed 3-submanifold in Q disconnecting its two boundary components, and let Σ_n be the corresponding diffeomorphic copy of Σ in $Q \times \{t_n\}$. It is clear that we can find some Σ_n arbitrarily close to the exotic end yielding a finite engulfing index $c(Y) \leq \beta_1(\Sigma)$, which is a contradiction.

□

Proposition 2.12. *Let L be a leaf diffeomorphic to Y. Then there exists an open saturated neighborhood U of L which is diffeomorphic to $L \times (-1,1)$ by a diffeomorphism which carries the bifoliation $\mathcal{F}$ and $\mathcal{N}$ to the product bifoliation. In particular, all the leaves of $\mathcal{F}_{|U}$ are diffeomorphic to Y.*

Proof. Since L is a proper leaf, there exists a path, $c : [0,1) \to M$, transverse to $\mathcal{F}$, with positive orientation and such that $L \cap c([0,1)) = \{c(0)\}$. Let U be the saturation of $c((0,1))$, which is a connected saturated open set and consider the octopus decomposition of $\hat{U}$ as described in Theorem 2.5. Clearly one of the boundary leaves of $\hat{U}$ is diffeomorphic to L because it is proper without holonomy and $c(0) \in L$. We identify this boundary leaf with L and extend the nucleus K so that the set $K' = \partial^\tau K \cap L$ is homeomorphic to K_Y. By Reeb stability, there exists a neighborhood of K' foliated as a product by $K_Y \times \{*\}$. Since $L \subset \partial\hat{U}$ has an end, there is an arm B_1 that meets L. The corresponding S_1 is diffeomorphic to $\mathring{X}_1$ and thus B_1 is foliated as a product (i. e., the suspension must be trivial). The union of a smaller product neighborhood of $L \cap B_1$ and the product neighborhood of K_Y meeting L gives a product neighborhood on the positive side of $X_1 \cup K_Y$. We can proceed in the same way for all the ends (which are finitely many), thus obtaining a product neighborhood on the positive side of $L \equiv K_Y \cup X_1 \cup \cdots \cup X_k$.

Proceeding in the same way on the negative side of L we can find the desired product neighborhood of L. Each leaf is clearly diffeomorphic to Y since the projection to L along leaves of $\mathcal{N}$ is a local diffeomorphism and bijective by the product structure. □

Let Ω be the union of leaves diffeomorphic to Y. By the previous Proposition this is an open set on which the restriction $\mathcal{F}_{|\Omega}$ is defined by a locally trivial fibration, so its leaf space is homeomorphic to a (possibly disconnected) 1-dimensional manifold. Let Ω_1 be one connected component of Ω.

Lemma 2.13. *The completed manifold* $\hat{\Omega}_1$ *is not compact.*

Proof. First we note that $\partial\hat{\Omega}_1$ cannot be empty, for otherwise all the leaves would be diffeomorphic to Y, hence proper and non-compact. It is a well known fact (see, e.g., [5]) that a foliation in a compact manifold with all leaves proper must have a compact leaf, for every minimal set of such a foliation is a compact leaf. Now suppose that $\hat{\Omega}_1$ is compact and let L be a leaf diffeomorphic to Y. Then the limit set of the exotic end of L contains a minimal set, which must be a compact leaf. This leaf, which we denote F, must be in the boundary of $\hat{\Omega}_1$. The holonomy of F has no fixed points (otherwise it would produce non-trivial holonomy on an interior leaf) and all the orbits are proper. Therefore the holonomy group of each boundary leaf must be isomorphic to $\mathbb{Z}$.

Let us observe that a smoothly periodic end has finite engulfing index, for periodicity allows us to find a copy of a disconnecting smooth closed 3-submanifold arbitrarily close to the given end.

Let h be the contracting map that generates the holonomy of F. Then there exists an open neighborhood $V \subset X_1$ of the exotic end of L where h is defined and induces an embedding $h : V \to V$ such that $\{h^n(V)\}$ $(n \geq 0)$ is a neighborhood base of the end (just by following the flow $\mathcal{N}$ in the direction towards F). Therefore V and h determine a smoothly periodic end in X_1 which is diffeomorphic to an end of $\natural_{i=1}^k \mathbf{R}$ (or to some finite end sum of $\mathbf{K}_t$'s), but this contradicts Theorem 1.7 since this exotic $\mathbb{R}^4$ has the same end as a punctured closed 4-manifold with intersection form $-kE_8 \oplus E_8$ which is not isomorphic to a diagonal form. □

Following the approach of Ghys [15], we have a dichotomy: the leaf space of $\mathcal{F}_{|\Omega_1}$, which is a connected 1-dimensional manifold, must be either $\mathbb{R}$ or S^1.

Proposition 2.14. *The leaf space of* $\mathcal{F}_{|\Omega_1}$ *cannot be* $\mathbb{R}$.

Proof. Since $\hat{\Omega}_1$ is not compact there exists at least one arm for its octopus decomposition. Let B_1 be such an arm that is diffeomorphic to $S_1 \times [0,1]$ via a diffeomorphism ϕ_1 carrying the vertical foliation to $i^*\mathcal{N}$. If the leaf space is $\mathbb{R}$, then $\phi_1(\{*\} \times (0,1))$ must meet each leaf in at most one point. Then the Trivialization Lemma 2.6 shows that the saturation of $\phi_1(\{*\} \times (0,1))$ is diffeomorphic to a product $L \times (0,1)$. Then the process of completing Ω_1 to $\hat{\Omega}_1$ shows that the product $L \times (0,1)$ extends to a product $L \times [0,1)$, so the boundary leaf corresponding to $L \times \{0\}$ must be diffeomorphic to Y, but this is a contradiction since leaves diffeomorphic to Y have to be interior leaves of Ω. □

Since Ω_1 cannot fiber over the line, it must fiber over the circle, but this is just the conclusion of Theorem 1, so its proof is complete.

3. The theory of levels and exotic structures

According to the previous section, each of our manifolds in $Y \in \mathcal{Y}$ that is realized as a leaf must be the fiber in a fibration of an open saturated set Ω_1 over the circle with diffeomorphic leaves as the fibers, and Ω_1 must spiral to somewhere in the ambient manifold. In this section we shall describe the theory of levels in C^{1+Lip} foliations and use it to show the following theorem. In the course of the proof of this theorem, we also prove Theorem 3, which is valid in all dimensions.

Theorem 3.1. *For any manifold $Y \in \mathcal{Y}$ there exists an uncountable subset $\mathcal{Y}_Y \subset \mathcal{Y}$ of (diffeomorphically distinct) manifolds homeomorphic to Y that are not diffeomorphic to any leaf at finite depth in a C^{1+Lip} codimension 1 foliation of a compact manifold.*

To begin the proof, let us fix $Y \in \mathcal{Y}$ and Ω_1 and consider the map $h : \Omega_1 \to \Omega_1$ which maps each point $x \in L' \subset \Omega_1$ to the first return point $h(x) \in L'$ along the transverse foliation $\mathcal{N}$ in the negative direction. This is well defined globally because each leaf has a neighborhood bifoliated as a product and the leaf space is the circle. This is the monodromy map which is an orientation-preserving automorphism of our exotic manifold L.

Next we recall level theory in codimension 1 foliations [5]. For this part we need extra regularity on the transverse changes of coordinates, which we assume to be at least C^{1+Lip} (i.e., the transverse coordinate changes have first derivatives that are Lipschitz functions).

Definition 3.2. A local minimal set of an open saturated set U is a non-empty set, closed in U and minimal for the relation of inclusion.

It is clear by Zorn's Lemma that a minimal set for $\mathcal{F}$ exists in a compact manifold, but in general it is false that every open saturated subset of $(M, \mathcal{F})$ contains a local minimal set.

Proposition 3.3. [5] *Let $(M, \mathcal{F})$ be a C^{1+Lip} codimension 1 foliation on a compact manifold. Then every open saturated set U contains a local minimal set. The union of the local minimal sets in a given U is closed in U.*

It follows that under this C^{1+Lip} hypothesis there exists a countable filtration $\emptyset \subset M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots \subset M$ where M_0 is the union of the minimal sets of $\mathcal{F}$ and, inductively, M_i is the union of M_{i-1} and the local minimal sets in $M \setminus M_{i-1}$.

Definition 3.4. A leaf belongs to the *level* k if it is contained in $M_k \setminus M_{k-1}$.

It is possible to have a leaf which does not belong to any finite level, but in this case it must be non-proper [5]. Therefore all the manifolds we have been considering as leaves are at finite levels, since they are proper leaves. For codimension 1 foliations, local minimal sets must be of one of the three following types: a proper leaf, the saturation of an open subset of a transverse Cantor set (these are called exceptional) or a saturated open set where all leaves are dense.

The theory of levels says that a leaf at level k accumulates on a finite number of local minimal sets at level $k-1$ and so on until we reach the level zero. When all the intermediate local minimal sets are proper leaves we say that our leaf has *depth* k. For a leaf L at depth k, we say that L' is a *closest leaf* to L if L accumulates on L' and there are no other leaves in the limit set of L which accumulate on L'. Note that a closest leaf need not be at depth $k-1$.

Lemma 3.5. *Let L be a leaf at depth k and let L' be a closest leaf to L. Then the action of the holonomy group of L' on L is infinite cyclic.*

Proof. Since L' is proper, Dippolito's theorem implies that there exists a small transversal T meeting L' in one point where a set of generators of the holonomy group of L' are defined. Since L' is a closest leaf to L and L is also proper, $L \cap T$ is a discrete set with exactly one accumulation point, which is $\{x\} = T \cap L'$. Since the foliation is transversely oriented we can work on only one side of T. It is clear that a generator $h \in \mathrm{Hol}(L', x)$ will have fixed points in $T \cap L$ arbitrarily close to x if and only if h fixes all the points in $T \cap L$ (otherwise $T \cap L$ would have an accumulation point different from x in T), and then the holonomy would be the identity on $T \cap L$, which is impossible, since L accumulates on L'. Otherwise $h_{|T\cap L}$ is determined by the image of $h(y) \neq y$ for some $y \in T \cap L$. There is a common divisor for the germs of this kind of maps, hence the action of $\mathrm{Hol}(L', x)$ in $L \cap T$ must be infinite cyclic. □

Proposition 3.6. [5] *If L is a leaf at finite depth, then its closure $\overline{L}$ is a finite union of leaves.*

Lemma 3.7. *Let L be a leaf at depth k. There exists a compact set $K \subset L$ such that each connected component $V \subset L \setminus K$ is contained in a normal neighborhood of a (unique) closest leaf L'.*

Proof. For each unbounded domain $V \subset L$ there exists a closest leaf L' where V is accumulating. Closest leaves are disjoint proper leaves at lower depth and there are only finitely many, so there exist normal neighborhoods (foliated by $\mathcal{N}$) pairwise disconnecting the closest leaves of L (but remark that closest leaves can share leaves on which they accumulate). Thus, there exists a compact set K such that $L \setminus K$ is contained in the union of these pairwise disjoint neighborhoods (because closest leaves are the first ones on which L accumulates). □

Next, we shall prove Theorem 3, which states that, up to diffeomorphism, there are only countably many open smooth manifolds which can be leaves at finite depth in a C^{1+Lip} codimension one foliation on a compact manifold.

Proof of Theorem 3. At depth 0 proper leaves are compact leaves, and there are only countably many smooth closed manifolds at this level (up to diffeomorphism).

Assume by induction that the set of smooth open manifolds which can lie at depth $i < k$ in any codimension 1 C^{1+Lip}-foliation is countable (up to diffeomorphism).

By Lemma 3.7 outside a compact domain K, a leaf L at depth k accumulates on a finite set of closest leaves and therefore there exists a compact set $K \subset L$ such that $L \setminus K$ is contained in a union of normal neighborhoods (foliated by $\mathcal{N}$) of these closest leaves where the generators of the holonomy group are defined. Each connected component of $L \setminus K$ accumulates on a closest leaf L' in a cyclic way, according to Lemma 3.5. Since the action of the holonomy of L' on L is determined by an element of the countable group $H^1(L', \mathbb{Z})$, it follows that, for a given choice of smooth structures on the closest leaves, each component of $L \setminus K$ admits at most countably many smooth structures compatible with them. They are obtained by lifting the structure of the closest leaf L', which is a leaf at lower depth.

If a given set of smooth structures on the closest leaves is changed to a diffeomorphic set of structures, it is clear that such a diffeomorphism lifts to a diffeomorfism on each connected component of $L \setminus K$. It follows that there is only a countable set of end-smooth structures up to diffeomorphism for leaves at depth k.

Fix an end-smooth structure E in $L \setminus K$. Let $\mathcal{L}(E)$ be the set of open smooth manifolds with the same smooth end structure. Up to diffeomorphism this set is countable since the family of smooth structures on compact manifolds with boundary is countable up to diffeomorphism.

If we change the end-structure E' in $L \setminus K$ to a diffeomorphic one then, for any element in $\mathcal{L}(E)$, there exists an element in $\mathcal{L}(E')$ which is diffeomorphic to it. Therefore, up to diffeomorphism, there exist at most countably many smooth open manifolds which can lie at depth k. This completes the induction. □

Since there are uncountably many smooth structures homeomorphic to Y in $\mathcal{Y}$, and by Theorem 3 only countably many can be leaves of finite depth, the proof of Theorem 3.1 is complete.

4. Transversely analytic foliations and exotic structures

Now for $Y \in \mathcal{Y}$, the manifolds in $\mathcal{Y}_Y$ are homeomorphic to Y and cannot be leaves at finite depth. In this section we shall prove Theorem 2 by showing that they cannot be leaves of transversely analytic foliations. Clearly, the properties of Ω_1 make it impossible for some intermediate local minimal set to be an open saturated set in which the leaves are dense. Since they cannot be leaves at finite depth, some intermediate local minimal set must be exceptional. To exclude this case, we shall use the beautiful theorem of Duminy.

Theorem 4.1 (Duminy, see [23]). *For a C^{1+Lip} codimension 1 foliation, the endset of a leaf passing through an accessible point of an exceptional local minimal set is homeomorphic to a Cantor set.*

It is time now for the unique result for transversely analytic foliations (see [6]) which will be used in this paper.

Definition 4.2. Let U be a connected open saturated set of a codimension 1 foliation on a compact manifold. We say that U is *trivial* at infinity if there exists an octopus decomposition of $\hat{U}$ such that the foliation on every arm is trivial (i.e., the total holonomy group in every arm is trivial). Note that this definition includes the case where $\hat{U}$ is compact.

Theorem 4.3 (Cantwell, Conlon, Hector, Duminy; see Lemma 3.5 and the subsequent Remark in [6]). *Let $\mathcal{F}$ be transversely analytic. Then every connected open $\mathcal{F}$-saturated set U is trivial at infinity.*

Proof of Theorem 2. According to Theorem 3.1, for any manifold $Y \in \mathcal{Y}$ there exists an uncountable subset $\mathcal{Y}_Y \subset \mathcal{Y}$ of manifolds homeomorphic to Y that are not diffeomorphic to any leaf at finite depth. As before, the leaves diffeomorphic to Y form an open saturated set, and we let Ω_1 be one of its connected components. By Theorem 4.3 the arms of the octopus decomposition of $\hat{\Omega}_1$ can be chosen to be trivial suspensions. Note that at least one arm exists by Lemma 2.13. Manifolds in $\mathcal{Y}_Y$ cannot lie at finite depth and so the boundary leaves of $\hat{\Omega}_1$ must accumulate on some exceptional local minimal set. In this situation each arm of $\hat{\Omega}_1$ must meet some gap T of the exceptional minimal set; the gap is an interval in a leaf of $\mathcal{N}$ whose interior is in the complement and whose endpoints are in the exceptional minimal set. Considering the two octopus decompositions of the saturation of T and of $\hat{\Omega}_1$, we can see that the arm B is embedded in an arm B' of the saturation of T, and there the foliation is again trivial by Theorem 4.3.

Since the suspension is trivial, every leaf L in Ω_1 meets the arm B' in one or more components, each of them homeomorphic to the intersection of the accessible leaf with the arm B'. Thus each connected component of $L \cap B$ has a Cantor set of ends (by Duminy's theorem) and a compact boundary since the boundary of the base manifold for the trivial suspension is a closed subset of the compact nucleus of the decomposition. As a consequence L has infinitely many ends, contradicting the fact that every manifold in $\mathcal{Y}$ has finitely many ends. □

Final comments

As far as we know, this work is gives the first insight into the problem of realizing exotic structures on open 4-manifolds as leaves of a foliation in a compact manifold. We express our hopes in the following conjecture, which we are far from proving, since it includes the higher codimension case and lower regularity assumptions, which are not treated in this paper. It is a goal for future research.

Conjecture 4.4. No open 4-manifold with an isolated end diffeomorphic to $\mathbf{R}$ or to $\natural^n\mathbf{R}$ is diffeomorphic to a leaf of a $C^{1,0}$ foliation of arbitrary codimension in a compact manifold.

Now let us say something about small exotica (those that embed as open sets in the standard $\mathbb{R}^4$). Small exotica are more interesting from a physical point of view since they support Stein structures (see e.g. [17]). There is a Taubes-type theorem for them based on the work of DeMichelis and Freedman [8] and with more generality in [29], but sadly it is not enough for us to adapt Lemma 2.13. In addition, there is no "Taylor-index" invariant and therefore the first part of our arguments, which shows that the leaf must be a proper leaf, fails for small exotic $\mathbb{R}^4$'s, although it works for punctured simply connected 4-manifolds obtained by removing finitely many points from closed manifolds not homeomorphic to S^4, since for these manifolds the argument is purely topological.

It is worth noting that if the smooth 4-dimensional Poincaré conjecture is false then it is easy to produce exotic $\mathbb{R}^4$'s which are leaves of a transversely analytic foliation. Consider $S^4 \times S^1$ with the product foliation, where S^4 has an exotic smooth structure, and insert a Reeb component along a transverse curve, for example $\{*\} \times S^1$. This can easily be done so as to preserve the transverse analyticity. The leaves would be exotic $\mathbb{R}^4$'s with a standard smooth structure at the end.

Finally we include a last remark. By a personal communication from J. Álvarez López, it seems possible that every Riemannian manifold with bounded geometry can be realized isometrically as a leaf in a compact foliated space. It is known [18] that every smooth manifold supports such a geometry, so it would follow as a corollary that every smooth manifold is diffeomorphic to a leaf in a compact foliated space. In particular this would be true for any exotic $\mathbb{R}^4$. However the transverse topology of this foliated space would in general be far from being a manifold. Anyway, this gives us some hope of finding an explicit description of exotic structures by using finite data: the tangential change of coordinates of a finite foliated atlas.

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