Teaching as an indicator of mathematical practices

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Abstract

In their study of the relationship between teaching and mathematical research, Belhoste and Schubring introduce a theoretical framework and a wealth of interesting case studies, highly substantiated by their study of primary sources. Following on their steps, I take teaching as a fertile indicator of the community's varying points of view about its practices, ranging from the interaction with other disciplines to the development of the strict mathematical corpus.

1 Introduction

As a working mathematician, I frequently compare myself as a keeper of a wonderful tradition. After millenia of contributions, mathematics is a diverse, powerful, critical body of knowledge. Its commentators add to the overall feeling of awe and curiosity. Philosophers ask about the role of logic in mathematics and in mathematicians' practices. Researchers in cultural studies consider its pervasive effect from pure science to the common man's attitude with respect to quantification and structure. The reciprocal influence among internal modalities — pure and applied mathematics, teaching and research — provides a fertile ground for epistemological debate: any aspect seems open ended.

It is thus not surprising that authors have different opinions about the interaction between teaching and the development of mathematical theory. I take as a starting point for my comments as a practitioner the juxtaposition of some of Belhoste's and Schubring's points of view. Belhoste's original article [2] was reviewed by Schubring [17], which then extends on the matter in [18]. One should not expect in this text the erudition of both authors, as displayed in [3] and [19].

A caveat: despite the fact of having Turing as one of my intellectual references, I will not refrain from using words as thinking, understanding, clarity. The inevitable imprecision should be compensated by the brevity of the presentation.

2 An overview

Following Kuyk ([12]), Belhoste and Schubring believe that the influence of teaching in the development of mathematical practices is underrated. In particular, Belhoste comments on the indifference towards the role of teaching in the historical development of mathematics. Schubring, in his subsequent articles, is enthusiastic about establishing a program dedicated to the reevaluation of the role of teaching.

A loose interpretation of teaching as the means of communication, transmission and diffusion of mathematical knowledge, may lead us to think that there is little left outside of teaching. Clearly, socialization in different forms is fundamental among researchers, students, practitioners in general: at the risk of trivializing the issue, one must specify relevant contexts. In a similar vein, there is too much encapsulated under the heading of mathematical research: the historiography of what is considered 'doing theory' is one of the most interesting aspects in history of the subject. Belhoste and Schubring agree that it is hard for the historian to separate production from the conditions of reproduction.

A satisfactory common ground to approach this ecology of concepts is achieved by the consideration of case studies, and both authors present expressive examples, to be commented later. In the words of Belhoste, teaching is a form of socializing knowledge, by converting it into normal science. Historically, teaching was fundamental in the creation of a career — the researcher in mathematics — and it was instrumental in the specification of different areas of mathematics through the compartimentalization of courses.

In order to bring specificity to the study of the rather ambiguous practices we call teaching and research, Schubring suggests a conceptual framework which naturally leads to interdisciplinary investigations. Two examples suffice to show the vastness of the project: the effect of ideology and religion on teaching, the reciprocal influence of practices of different subjects and their effect on society.

3 Textbooks: from Babylonia to Bourbaki

A wealth of very well chosen examples is presented by both authors ([2],[17],[18]). The purpose of my very schematic presentation is to indicate some relevant aspects to be followed along the text.

Schubring comments on Proust's work on Babylonian clay tablets ([15]). She sorts some 2000 pieces and distinguishes among exercises for young scribes, the occasional tables, problems followed by their resolution, and problems which seem to be open, expecting consideration. The whole gamut of practitioners is covered, from beginning students to scholars. Jumping to the nineteenth century, Schubring quotes de Tracy, (1801): the preparation of textbooks requires creativity for the appropriate presentation of known facts, a kind of research by itself.

Cauchy in his years at the École Polytechnique attached research to teaching by establishing levels of rigor, a pedagogical (ideological, according to Schubring) requirement. Dedekind in 1872 points out to the lack of foundations which led to his *Stetigkeit und irrationale Zahlen*. Meyer wrote about sets (1885) while Cantor was still perfecting his theory. From [18],

> ... Cantor's ideas were for Meyer ... only the trigger for developing fundamental concepts developed already since a long time by mathematics teachers. He did not present the concept of set as something new, but as belonging to a tradition going back to the ancient Greeks.

The debate about teaching through set theory seems to begin here. According to Schubring, Klein was strongly against it: it required further understanding and did not seem fertile. Again, from [18],

> ... [the subject seemed far] from having matured to the point of having induced and intradisciplinary process of integration and restructuration. The concepts of set theory did non (yet) provide new elements for mathematics.

Schubring acknowledges the importance of Beltrami's text of 1868 to the acceptance of non-Euclidean geometry by the mathematical community as a whole. The concrete model presented in the text is exemplary — how to distinguish among research, clarification, the need to convince the skeptics? Schubring draws attention to another author on the subject, Wagner, which, like Meyer, makes an effort to combine teaching and rigor. Significantly, his book has two prefaces, for colleagues and for students.

Belhoste ascribes the origins of Bourbaki's monumental project to the desire of an alternative text to Goursat's analysis book. In his conclusion, Schubring points out to an interesting issue:

> ... it is not possible to close our eyes to the fact that dogmatic, formalizing impulses for the development of science have also emerged from the school: in a fundamentalist exaggeration of the search for firm foundations.

All such examples are significative, and new ones will surely come up with the expansion of research on these lines. But instead of searching for common trends, I prefer to consider them as indicators of different contexts.

Another set of examples of how subtle is the line between teaching and research is the presence of mathematical results which first appear as exam questions. Spivak ([22]) points out that Stoke's theorem, originally quoted in a letter from Kelvin to Stokes, is also question 8 on the Smith's Prize Examination for 1854. Bost and Mestre ([4]) present a result by Richelot on a generalization of the arithmetic-geometric-mean for surfaces of genus 2 with a similar fate. One is also reminded of more or less apocriphal stories about (non Babylonian) students who solve open problems thinking that they were homework assignments. The Fáry-Milnor theorem, according to Milnor himself, is not such an example. But, according to Kuperberg ([12]),

> ... Dantzig was in graduate school in statistics, it was 1939, he came in late to class, he saw two open problems on the board and thought they were homework. The professor was Neyman, who must have been a bit impenetrable to his students, because he didn't tell Dantzig for six weeks what had really happened.

4 Professionalization

Both Belhoste and Schubring expand on the importance of teaching for the characterization of a mathematical profession, and again the issue has a number of different aspects. Indeed, there is a substantial difference between the importance given to mathematics by those who study music in the quadrivium and the fact that the State needs administrative specialists, not to mention statisticians. Cauchy is certainly an interesting case study for the rise of the public servant, but the teaching of applied mathematics was already present in universities.

There is a relevant curve to follow in the historical process: university calculus seems to be disconnecting from the demands of the professional world. Other university departments provide mathematical material which correlates better with a computerized world of large scale modeling and simulation. Is there a difference in what is taught between private and public contexts?

Thurston [25] provides a wonderful case study of the economics of mathematics research. His results in foliation theory were so good that students were suggested to drop the subject. The influence of trends in mathematics looks small when compared to physics.

The necessity of an interdisciplinary point of view is clear: in some situations, mathematics gets stimulated and in others, confined. Schubring considers the employment of mathematical teaching to religious demands, a rich subject. One is tempted to extrapolate: mathematics seems to be more accepted as a tool for higher efficiency in the armies than in other fields of the government.

5 An interdisciplinary project

As Schubring points out, the interaction of mathematics with other cultural activities is an interdisciplinary subject: in order to understand mathematical production, one must see it in relation with other social subsystems. The scale of such a project is immense, the influence of teaching encompasses the presence of mathematics in our intellectual life.

The issue is unavoidable and Schubring states it clearly [17]: there was never autonomy for mathematics (and here he is emphasizing the teaching activities). In secondary school, mathematics is a concurring subject, together with other values conveyed by ideology. In universities, it negotiates with other disciplines. Research centers before 1800 would concentrate in teaching (and nowadays such institutions require substantial support from the State).

Below, three lines of investigation are barely sketched.

5.1 Mathematics as an international activity

Schubring proposes as a research subject the confluence of local projects leading to an international mathematical community [17]. Following Luhmann and Stichweh ([13], [24]), he takes communication as the elementary act of science, foundational to teaching, learning and doing research. Communication requires a common language and a shared culture, and institutions and states become protagonists of the supranational interchange.

The original ideas shaping mathematics are so old that the temptation of treating them as ahistorical is unavoidable. A schoolboy learning Pythagoras' theorem (a beautiful example of incorporation of mathematical knowledge of yet older times) probably has no idea of Greek history, or philosophy.

5.2 Collective and individual practices

There is no need for agreement on the matter: interesting examples of both positions are abundant. A recent improvement on mathematical communication is the presence of very active blogs, like Tao's or Mathematics Stack Exchange.

Thurston's presentation of his own mathematical experience borders on the dramatic [25]. He begins emphasizing the introspection required for his understanding of foliation theory, which included intensive computer programming:

> The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical's community standard of valid proofs.[1]

Thurston then describes the necessary effort for spreading his knowledge. His statement is an inextricable merging between teaching and research: one does mathematics to obtain understanding and it takes energy to convey not only proofs of new results, but 'the ways of thinking about the same mathematical structure'. Belhoste [2] quotes the many contributions to elliptic function theory in the mid-nineteen century as teaching landmarks, and Schubring's opinion [17]

[1]The interaction with computers is already a subject for the history of mathematical practices. Different reactions within university departments are leading to a radical design of professional and teaching standards.

is that of a practitioner — this is pretty much what research is about, the natural development of concepts [2]. Indeed, the community assigns different values to the verification of a claim and its absorption in the corpus.

There is a situation in the boundary between private and public practices which deserves more attention. A common place within the community is how much there is to learn from a teacher who thinks while giving his class (as opposed, say, to someone who just follows a prepared text). There is some truth to it: errors should be part of teaching, there are few occasions to appreciate a specialist in intellectual action. There are also familiar opposing arguments: students (and teacher) may get confused, intimidation may pop up at any time [3]. But perhaps there is more to it: some people might prefer to think in public — a positive form of vanity stimulates the capacity of verbalizing a subject.

In a somewhat reciprocal fashion, one is led to think if genuine collective study would eliminate the case studies referred in [21].

5.3 Abstracting, forgetting, getting to the point

A prehistoric man who owns a few sheep finds a stone which reminds him of one of them. At some point, he may have gathered a collection of such amulets, representations of the animals which are so dear to him. It must have taken a long time for a collection of rocks to become an indicator of the size of a flock, possibly the first application of a forgetful functor. One wonders: how many squares has to be drawn side by side in a vase until *the* square becomes *a* square? Coherently, the usual references to the origins of counting do not treat these issues.

Abstracting is not natural: it took a figure of authority like Hilbert to make the mathematical community get used to the fact that some initial words are devoid of meaning when building up an axiomatic system, and this certainly helped the acceptance of non-euclidean geometries [7]. Why is a circle not a straight line for a topologist? [4] Why are $\mathbb{R}^2$ and $\mathbb{R}^3$ different? One understands that geometry was a good subject to practice argumentation ([18]), being less polemical than religion or politics, but there is more: the discussion may end. Some intellectual practices thrive on open-endedness: forgetting may become suspicious, one is responsible for whatever is being discarded.

[2] One is reminded of Besicovitch, 'a mathematician's reputation rests on the number of bad proofs he has given'.

[3] Recall the familiar story attributed not only to some mathematicians, but also to physicists. The teacher states, 'This is trivial' and a student asks why. The teacher leaves the room, and when he comes back, makes it clear, 'yes, it's trivial' and goes on with his class.

[4] And still 'you must remember this... the plane is not a disk...' (As z goes by).

6 Missing opportunities in teaching

Why is it that in mathematics we do not acknowledge epistemological bangs, but just the occasional whimper of increasing consensus?[5] Some exhaustively studied episodes in the history of mathematics deserve more recognition (percolation seems to be the right word). The issue affects the contents of our courses, and provide a significative indication of a certain hierarchy of relevance, in which conceptual aspects are less important than more pragmatic practices.

Clearly some fundamental problems have ceased to be so: the community, from students to researchers, do not seem to worry anymore about the distinction between numbers and quantities, which is usually considered as being resolved by Stevin ([14],[17]), or by the omnipresent rule of signs [7]. I present a couple of issues which may still intrigue a current practitioner.

6.1 Optimal design

Most calculus students are confronted with the computation of the path of light joining source and target hitting a mirror along the way. What is originally a simple problem in finding a minimum opens up to the metaphysical issues associated to Maupertuis' principle: very informally, nature behaves in the best possible way. I was impacted when entering university by this spectacular evidence of intelligent design much before the spreading of the creationist debate. Teachers both in mathematics and physics did not come up with satisfactory answers. Years later, I learned that I was in great historical company: intellectual comfort came by accident, through Ekeland's wonderful book [5]. To say the very least, we miss an opportunity of presenting one of the great concepts of physics — the idea of averaging over all paths, aka the Feynman integrals ([6]).

And then things turn upside down: instead of brilliant photons, they are so stupid (or homogeneous in their behavior) that it is the appropriate average of all their choices which makes one think that an optimal decision was taken. The point of view is well known in social sciences under the catchphrase 'the wisdom of the crowds' [23]. In optimization and signal processing, the (hard) search for optimal behavior is frequently replaced by sampling a typical object, which sometimes performs in an almost optimal fashion.

6.2 Instantaneous interaction

Consider the criticism against Newton for making use of instantaneous gravitational interaction. Some of his contemporaries filled the universe with stuff that would somehow transmit the forces. Newton discarded the issue, defending his position by simply presenting the output of his calculations. Why shouldn't a

[5] Azzouni [1] refers to the benign fixation of mathematical practice, which he takes to be an almost unique attitude towards consensus, specific to the mathematical community.

contemporary student be less intrigued than Descartes? What is offered in return for this curiosity? Riemann's ideas, which led to what Wheeler would call geometrodynamics — geometry induces physics — are a form of esoteric knowledge, a privilege of few. As far as intellectual stimulation goes, almost nothing compares to the idea that it is the geometry of some enlarged space which makes you think that a planet is pulling down an apple.

6.3 Method

The flatness of the Earth is out of fashion, we have seen the photos of a round planet, which the fundamental implication that we live in a bounded surface with no boundaries. Still, for most people, a bounded universe would require a wall enclosing it. Concepts think for us: in this case we should get used to three dimensional compact manifolds. And then, one might wonder, what about orientation? I do not think that teaching does a good job in making explicit the power of well chosen concepts.

Computers generated an abundance of theorems in plane geometry which seem to be irrelevant to what is considered the body of knowledge on the subject. Indeed, the presentation of a mathematical subject is hardly a thick description, in the anthropological sense. It is more like the description of a city by first indicating its main spots, from which one may provide more detailed information in order to reach any other point. Mathematicians frequently talk about geometrizing mathematics: Euclid's axioms are taken to higher dimensions, to discrete structures. But when we tell students that two distinct points determine a unique line in 200 dimensions, we should make it clear: this is not a visual fact — or is it? We are missing epistemological labs.

Learning is not simple, and the history of mathematics is a collection of impressive difficulties. One of the mantras among practitioners is the fact that the Greeks came up with the axiomatic method as a fertile method for the presentation and generation of mathematics. There is much to learn from verifying the statement throughout the millenia, but also to understand why it is so strongly believed. I outline two examples.

Negative numbers and virtuality

One could hardly start from a better place than Glaeser's ' Épistemologie des nombres rélatifs' [7] for a presentation of the history of the rule of signs (the product of two negative numbers is a positive number). For a practitioner, the changes of opinion among major mathematicians provide a strong argument for knowing more about the history of the subject. The issue seems to have been resolved satisfactorily to our current standards in a text by Hankel's ([9]) on complex numbers: according to Glaeser, Hankel performs the passage from the concrete to the formal level. But there is more, a problem which is presented almost as an understatement by the author:

La révolution accomplie par Hankel consiste à aborder le problème dans une toute autre perspective. Il ne s'agit plus de déterrer dans la Nature des exemples pratiques qui 'expliquent' les nombres relatifs sur le mode métaphorique. Ces nombres ne sont plus découverts, mais inventés, imaginés.

On the one hand, there was no satisfactory metaphor: what is the product of two debts, one would react at a naive proposal? On the other, there was no answer from the Heavens. At this point, the standard practitioner should be intrigued — the axiomatic method has been disregarded for two millenia.

There are psychological aspects behind the change of attitude. Alas, it is not Nature which validates mathematics: validation follows from the axioms which we choose, rules devoid of any transcendence. The virtual world — Turing's paradise — should be the stomping ground of the mathematical community. Hardy should get used to the idea that intelligence services live out of his research.

We are reminded of Rota's description of the double life of mathematics ([16]):

> ... The facts of mathematics are as useful as the facts of any other science. No matter how abstruse they may first seem, sooner or later they find their way back to practical applications... Axiomatic exposition is indispensable in mathematics because the facts of mathematics, unlike the facts of physics, are not amenable to experimental verification.

Noneuclidean geometry

Few practitioners realize that there was no clear agreement to what a noneuclidean geometry meant. Gray's text is a short, incisive account [7], presenting the difficulties and the required change of attitudes. Spherical geometry was out: two lines should not enclose a bounded area. Because of Kant's influence, the right axioms had to fit with our innate conception of space. This is in sharp contrast to our current practice: different geometries simply depend on different starting points. Gray stresses the importance of symbolic language:

> ...by casting geometry into trigonometrical formulas, Bolyai and Lobachevsky both produced a language for analysing the behaviour of lines that evades the weight of tradition.

For a layperson, the underlying conceptual revolution is surprisingly recent:

> The far more radical step of denying mathematical terms any meaning and relying completely on formal rules of inference was taken by Peano and, independently by Hilbert.

The fertility of the new, looser interpretation of the undefined concepts was evident: one could do geometry in so many different contexts, for example, ranging from finite fields to Banach spaces.

No-go theorems, fertile imprecision

Some important ideas in physics are prohibitions: there are no monopoles, there is nothing faster than light, certain properties are conserved. Are they theorems? Not really, in the sense that few practitioners would know the underlying hypotheses of such claims. Perhaps a better contextualization would be saying that such statements are true in the light of all the facts we believe.

From a mathematical point of view, the attitude is an acknowledgement of coexisting sets of axioms. This happens also among practicing mathematicians: the Cayley-Hamilton theorem for complex matrices may be proved using Cauchy's theorem for analytic functions, and with a little syntactic push reminiscent of model theory, the argument extends to commutative rings with unity.

Moreover, a physicist is willing to occasionally change the axioms while research advances. An annoying corollary arises in occasional interactions between mathematicians and physicists. The mathematician presents his results on a specific issue (say, the Schrödinger equation for the hydrogen atom; the example is taken from [20]) and the physicist immediately asks what happens in a slightly different formulation. Indeed, the equation is a model, built up from hypotheses and simplification and is thus amenable to fudging. Still, an understanding of simple cases is one of the best contributions a mathematician can offer.

New students, new teachers

The idea of what constitutes an elementary subject is an interesting example. One may think that the elementary material is that which is taught to beginners. Logicians and participants in mathematical olympiads see it differently: a more elementary subject has less pre-requisites. Anachronistically, researchers like Frege and Boole seem like unconsciously preparing material for the dumbest student, the computer. Perhaps the community will soon be talking about machine teaching hand in hand with machine learning. [6]

7 A personal recollection

Perhaps the hardest part in the study of the ties between teaching and research practices is the identification of soft fossils, the so many episodes which are not documented in a stable fashion.

The following Zen story is traditional. A monk shows a fan to three students and asks, 'what is this?' The first replies, 'it's a fan', and the monk is not pleased. He shows the fan to the second student, which picks the fan from the master's hands and starts waving himself. The master is still not satisfied and shows the fan to the third student, which takes the fan and uses it to scratch his back.

This story made more sense to me during my PhD years at the Courant Institute. As a student, I was convinced that powerful mathematics came from the knowledge of better theorems. This changed after being exposed to a number

[6] At the risk of being criticized for specism.

of situations in which the masters would scratch their backs with theorems I knew. After a few examples, I felt entitled, more, I felt invited to do the same. This exercise of intellectual freedom was the most important lesson of my graduate years. Students are surprised when they realize that there are many addition tables, many metrics, all triangles have a straight angle.

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