A theorem on cartography by Lagrange, the complex exponential function and the Euclidean plane as a paraboloid in the light cone

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Mathematical cartography has had a prominent rôle in the early development of differential geometry. It was Euler who first gave a proof of a fact that map-makers had long realized in practice: there are no ideal map projections of an open subset of the sphere onto an open subset of the plane. By an *ideal* map projection we mean a map that preserves lengths, and hence all other relevant geometric properties, as angles and areas. In other words, no open subset of the sphere can be *isometrically* mapped onto an open subset of the plane. This was later clarified by Gauss Theorem Egregium, which revealed the fundamental fact that preservation of *curvature* is an obstruction for the existence of such a map between any two surfaces.

Gauss also realized that preservation of angles was then the crucial property of a cartographic map, as he pointed out in a letter to Hansen from December 11, 1825:

You are quite right that the essential condition in every map projection is the infinitesimal similarity; a condition that should be neglected only in very special cases of need.

In contrast to isometric maps, Gauss proved that there are no obstructions to construct (local) analytic conformal (angle-preserving) maps from any analytic surface into the plane. This also holds in the smooth case, but a proof of this fact only appeared much later.

Another useful feature of a cartographic map of the sphere is that meridians and parallels be represented by arcs of circles or straight lines. This has led Lagrange, as early as 1779, to pose (and solve) the problem of determining all *tracées geographiques* (angle-preserving mappings into the plane) of an open subset of the sphere such that meridians (or parallels) are mapped into (arcs of) circles or straight lines.

In this note we give a proof of Lagrange's theorem. Along the way we discuss basic facts on conformal (angle-preserving) mappings, including the steoreographic and Mercator map projections of the sphere, explain how the complex exponential map shows up in connection with them, and introduce the model of Euclidean plane as a paraboloid in the light cone of Lorentz space. We use the latter in order to give a simple proof of the characterization of ortogonal families of circles and straight lines in the Euclidean plane, which is needed in the proof of Lagrange's theorem.

1 Conformal mappings

A map $f\colon S_1 \to S_2$ between two surfaces in $\mathbb{R}^3$ is said to be *conformal* if it preserves angles. More precisely, given two smooth curves $\alpha, \beta\colon I \subset \mathbb{R} \to S_1$ with $0 \in I$ and $\alpha(0) = p = \beta(0)$, the angle between $v = \alpha'(0)$ and $w = \beta'(0)$ must coincide with the angle between $(f \circ \alpha)'(0) = df(p)(\alpha'(0))$ and $(f \circ \beta)'(0) = df(p)(\beta'(0))$. Setting $T = df(p)$, this condition may be written as

$$\frac{\langle Tv, Tw\rangle}{|Tv||Tw|} = \frac{\langle v, w\rangle}{|v||w|} \quad \text{for all } v, w \in T_pS_1. \tag{1}$$

It is easily seen that a linear map $T\colon T_pS_1 \to T_pS_2$ satisfies (1) if and only if there exists $\lambda = \lambda(p) \in \mathbb{R}$ such that

$$\langle Tv, Tw\rangle = \lambda\langle v, w\rangle \quad \text{for all } v, w \in T_pS_1. \tag{2}$$

In other words, a conformal map is an *infinitesimal similarity*. The function $p \mapsto \lambda(p)$ is easily seen to be smooth, and is called the *conformal factor* of f. Notice that (2) is satisfied if and only if for some (and hence for any) orthogonal basis $\{v, w\}$ of T_pS_1 it holds that $T(v)$ and $T(w)$ are orthogonal and satisfy

$$\frac{|Tv|}{|v|} = \frac{|Tw|}{|w|}.$$

In particular, a map $f = (u, v)\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is conformal if and only if the vectors $df(e_1) = (u_x, v_x)$ and $df(e_2) = (u_y, v_y)$ are orthogonal and have the same length. Therefore, either $u_x = v_y$ and $v_x = -u_y$ or $u_x = -v_y$ and $v_x = u_y$. In other words, a map $f = (u, v)\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is conformal iff it is either holomorphic or anti-holomorphic, according as it preserves or reverts orientation.

The holomorphic map $f(z) = 1/z, z \neq 0$, has one important additional property: it maps circles and straight lines into circles or straight lines. This can be seen as follows: the equation of any circle or straight line in $\mathbb{R}^2$ can be put into the form

$$A|z|^2 + B\bar{z} + \bar{B}z + C = 0, \tag{3}$$

where A and C are real and $|B|^2 \geq AC$, the case of a straight line corresponding to $A = 0$. Dividing this equation by $|z|^2$ gives

$$C|1/z|^2 + B(1/z) + \bar{B}(1/\bar{z}) + A = 0,$$

thus $w = 1/z$ satisfies $C|w|^2 + Bw + \bar{B}\bar{w} + A = 0$, which is an equation of the same type as (3).

The map $I(z) = 1/\bar{z} = \overline{f(z)}$, $z \neq 0$, is therefore an anti-holomorphic map that takes circles and straight lines into circles or straight lines. It can be written as $I(z) = z/|z|^2$, so $I(z)$ is characterized geometrically as the unique point in half-line line joining the origin to z such that the distances $|z|$ and $|I(z)|$ from z and $I(z)$, respectively, to the origin, satisfy $|z||I(z)| = 1$. It is called the *inversion* with respect to the unit circle. More generally,

$$I_{r,z_0}(z) = z_0 + r^2(z - z_0)/|z - z_0|^2$$

is the inversion with respect to the circle of radius r centered at z_0. Check that I_{r,z_0} leaves lines through z_0 invariant, takes circles through z_0 onto lines not through z_0 and conversely, and circles not through z_0 onto circles not through z_0.

Another holomorphic function that will be of importance for us is the complex exponential function

$$\exp(x + iy) = e^x(\cos y + i \sin y).$$

It maps each strip $U_\theta = \mathbb{R} \times (\theta, \theta + 2\pi)$ diffeomorphically onto the complement $\mathbb{C} \setminus L_\theta$ of the half-line through the origin that makes an angle θ with the x-axis. The inverse *Log* of $\exp|_{U_{-\pi}}$ is called the *principal branch* of the complex logarithmic function. Notice that exp maps coordinate lines $x = x_0$ onto circles of radius e^{x_0} and coordinate curves $y = y_0$ onto straight lines through the origin making an angle y_0 with the x-axis.

2 Mercator and stereographic projections

There are two famous conformal maps of the sphere. The first one is the stereographic projection, defined as follows. Take a unit sphere $\mathbb{S}^2$ in Euclidean space centered at the origin. Then associate to each point $(x, y, z) \in \mathbb{S}^2 \setminus N$, $N = (0, 0, 1)$, the intersection of the half-line joining N to (x, y, z) with the xy-plane. This is given by

$$\pi(x, y, z) = (0, 0, 1) + t(x, y, z - 1),$$

where t is determined so that $1 + t(z - 1) = 0$. So $t = 1/(1 - z)$, which gives

$$\pi(x, y, z) = \frac{1}{1 - z}(x, y).$$

If we parameterize $\mathbb{S}^2$ by the latitude and longitud φ and θ, respectively, by means of the map

$$X\colon (0, 2\pi) \times (0, \pi) \to \mathbb{S}^2, \quad X(\varphi, \theta) = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi),$$

we have

$$\pi(X(\varphi, \theta)) = \frac{\cos\varphi}{1 - \sin\varphi}(\cos\theta, \sin\theta). \tag{4}$$

The other famous conformal map of the sphere is Mercator's projection. It is defined by

$$M(X(\varphi,\theta)) = (h(\varphi),\theta),$$

with h chosen so as to make it conformal, with $h(0)=0$. We must have

$$\frac{|dM(X_\varphi)|}{|X_\varphi|} = \frac{|dM(X_\theta)|}{|X_\theta|},$$

that is,

$$h'(\varphi) = \frac{1}{\cos\varphi}.$$

Therefore,

$$M(X(\varphi,\theta)) = (log(\cos\varphi/(1-\sin\varphi)),\theta). \tag{5}$$

It is amazing that Mercator, a Flemish-Dutch cartographer, knew this map already in 1569, quite some time before Calculus was invented!

Comparing (4) and (5), we see that

$$\exp(M(X(\varphi,\theta)) = \pi(X(\varphi,\theta)),$$

so the two projections differ by the complex exponential function!

3 Lagrange's theorem

We are now in a position to discuss the problem posed by Lagrange, namely, to determine all conformal mappings into the plane of an open subset of the sphere such that meridians (or parallels) are mapped into (arcs of) circles or straight lines.

Let $M\colon U \subset \mathbb{S}^2 \to \mathbb{R}^2$ denote the Mercator projection and let $\pi\colon V \subset U \subset \mathbb{S}^2 \to \mathbb{R}^2$ be another conformal map satisfying the condition in Lagrange's problem. Then

$$f = \pi \circ M^{-1}\colon W := M(U) \to \mathbb{R}^2$$

is a conformal map that takes one family of coordinate lines into (arcs of) circles or straight lines. The following result [**To**] characterizes all conformal maps $f\colon U \to \mathbb{R}^2$ with this property, thus showing that all solutions of Lagrange's problem are given in terms of such maps by

$$\pi = f \circ M.$$

Theorem 1 *Let $f\colon U \to \mathbb{R}^2$ be a conformal map defined on the connected open subset $U \subset \mathbb{R}^2$. Assume that one family of coordinate curves is mapped by f into a family of (pieces of) circles or straight lines. Then there exist an inversion I with respect to a circle of unit radius, a translation L and a composition H of a dilation, a translation and reflections in the coordinate axes and the line $y=x$, such that $f = I \circ L \circ \exp \circ H|_U$, or else f is such a composition with possibly some of its components replaced by the identity map.*

Proof: We give an alternate proof to that in [**To**]. The first step is to show that also the other family of coordinate lines is mapped by f into a family of circles or straight lines. This is a consequence of the following

Lemma 2 *Let $f\colon U \to \mathbb{R}^2$ be a conformal map defined on the connected open subset $U \subset \mathbb{R}^2$. Denote by $k_1(x_0, y_0)$ (resp., $k_2(x_0, y_0)$) the curvature at x_0 (resp., y_0) of the curve $x \mapsto f(x, y_0)$ (resp., $y \mapsto f(x_0, y)$). Then $(k_1)_x(x_0, y_0) = -(k_2)_y(x_0, y_0)$.*

Proof: We have

$$k_1(x, y) = \frac{u_{xx}v_x - v_{xx}u_x}{(u_x^2 + v_x^2)^{3/2}}.$$

Using the Cauchy-Riemann equations, we get

$$u_{xx}v_x - v_{xx}u_x = v_{xy}v_x + u_{xy}u_x = \frac{1}{2}(u_x^2 + v_x^2)_y,$$

hence $k_1 = -(E^{-1/2})_y$, with $E = u_x^2 + v_x^2$. Similarly,

$$k_2(x, y) = \frac{u_{yy}v_y - v_{yy}u_y}{(u_y^2 + v_y^2)^{3/2}} = \frac{-v_{yx}v_y - u_{yx}u_y}{(u_x^2 + v_x^2)^{3/2}} = (E^{-1/2})_x \quad ∎$$

Therefore, the images by f of the families of coordinate lines give two families of circles or straight lines, any element of each one is orthogonal to every element of the other. We now use the following classical fact [**Da**], a proof of which is given in the next section:

Proposition 3 *Let two families of straight lines and circles, each of which with at least two elements, have the property that every member of one family be orthogonal to every member of the other. Then either they are orthogonal families of parallel lines, or one of them is a family of concentric circles and the other a family of straight lines through the common center, or there exists an inversion that maps them into families of one of those two types.*

Compose f with an inversion I given by Proposition 3. Assume that $I \circ f$ maps the coordinate lines into families of straight lines and circles of the second type. Now compose $I \circ f$ with a translation T that takes the common center of the circles of one of the families into the origin. Let W be the domain of the principal branch Log of the complex logarithmic function and set $V = (T \circ I \circ f)^{-1}(W)$. Then $\mathrm{Log} \circ T \circ I \circ f|_V$ is a conformal map that, after possibly a further composition with a reflection in the line $y = x$, takes coordinate lines into coordinate lines with respect to the same coordinate. It is easily seen that such a map H is, up to a translation and reflections in the coordinate axes, a dilation by a nonzero constant. Setting $L = T^{-1}$, we obtain that

$$f|_V = I \circ L \circ \exp \circ H|_V.$$

By analyticity, f must coincide with $I \circ L \circ \exp \circ H$ on U. ∎

4 Euclidean plane as a paraboloid in the light cone

If $\mathbb{R}^4$ is endowed with a Lorentz scalar product

$$\langle\langle v, w\rangle\rangle = -v_0w_0 + v_1w_1 + v_2w_2 + v_3w_3,$$

for $v = (v_0, v_1, v_2, v_3)$ and $w = (w_0, w_1, w_2, w_3)$, then it becomes the 4-dimensional *Minkowski space*, and is denoted by $\mathbb{L}^4$. A vector $v \in \mathbb{L}^4$ is said to be *space-like*, *light-like* or *time-like* according as $\langle\langle v, v\rangle\rangle > 0$, $\langle\langle v, v\rangle\rangle = 0$ or $\langle\langle v, v\rangle\rangle < 0$, respectively. The same terminology is used for a subspace $V \subset \mathbb{L}^4$, depending on whether the restriction of $\langle\langle\ ,\ \rangle\rangle$ to V is positive-definite, degenerate (i.e., $V \cap V^\perp \neq \{0\}$) or Lorentzian, respectively. The set of light-like vectors

$$\mathbb{V}^3 = \{p \in \mathbb{L}^4 \colon \langle\langle p, p\rangle\rangle = 0\}$$

is called the *light cone* of $\mathbb{L}^4$. The intersection

$$\mathbb{E}^2 = \mathbb{E}^2_w = \{p \in \mathbb{V}^3 \colon \langle\langle p, w\rangle\rangle = 1\}$$

of $\mathbb{V}^3$ with the affine hyperplane $\langle\langle p, w\rangle\rangle = 1$ is a model of the Euclidean plane for any $w \in \mathbb{V}^3$. Namely, fix $p_0 \in \mathbb{E}^2$ and a linear isometry $A \colon \mathbb{R}^2 \to \{p_0, w\}^\perp$. Then the map $\Psi = \Psi_{p_0,w,A} \colon \mathbb{R}^2 \to \mathbb{E}^2 \subset \mathbb{L}^4$ given by

$$x \in \mathbb{R}^2 \mapsto p_0 + A(x) - (1/2)|x|^2 w$$

is an isometry. This follows by computing

$$d\Psi(x)X = A(X) - \langle X, x\rangle w \ \text{ for all } x, X \in \mathbb{R}^n, \tag{6}$$

which gives

$$\langle\langle d\Psi(x)X, d\Psi(x)Y\rangle\rangle = \langle X, Y\rangle$$

for all $x, X, Y \in \mathbb{R}^2$.

4.1 The space of circles

Circles in Euclidean space $\mathbb{R}^2$ have a neat description in its model $\mathbb{E}^2$: let $\alpha \colon \mathbb{R} \to \mathbb{R}^2$ be a unit speed parametrization of an oriented circle or straight line $C \subset \mathbb{R}^2$ with (constant) curvature k and let $\{t(s), n(s)\}$ be the Frenet frame of α. Differentiating the map $\rho \colon \mathbb{R} \to \mathbb{L}^4$ given by

$$\rho(s) = d\Psi(\alpha(s))n(\alpha(s)) + k\Psi(\alpha(s))$$

and using (6) we get

$$\rho' = A(n') - \langle n', \alpha\rangle w + kA(t) - k\langle t, \alpha\rangle = 0$$

by the Frenet formula $n' = -kt$. Hence ρ is a constant unit space-like vector $v \in \mathbb{L}^4$ with $\langle\langle \Psi(\alpha(s)), v \rangle\rangle = 0$ for all $s \in \mathbb{R}$. It follows that $\Psi(C) = \mathbb{E}^2 \cap \{v\}^\perp$, and from now on we write $C = \mathbb{E}^2 \cap \{v\}^\perp$ for short. Observe that C is a straight line iff $0 = k = \langle\langle v, w \rangle\rangle$. Notice also that changing the orientation of α amounts to changing the unit normal vector field n by a sign, and hence the corresponding curvature k, which makes the unit space-like vector v also to change its sign. Thus, unit space-like vectors in $\mathbb{L}^4$ are in one-to-one correspondence with *oriented* circles and straight lines of $\mathbb{R}^2$, hence the space of oriented circles and straight lines of $\mathbb{R}^2$ is naturally identified in this way with *de Sitter space* $\mathbb{S}^3_1$ of all unit space-like vectors of $\mathbb{L}^4$.

The relative position of a pair (C_1, C_2), with C_i a circle or straight line for $i = 1, 2$, has a simple description in this model: for instance, if $C_1 = \mathbb{E}^2 \cap \{v_1\}^\perp$ and $C_2 = \mathbb{E}^2 \cap \{v_2\}^\perp$ are circles, then $C_1 \cap C_2 = \mathbb{E}^2 \cap V^\perp$, where V is the subspace spanned by v_1 and v_2. Thus $C_1 \cap C_2$ consists of two points, one single point or is empty according as V is space-like, light-like or time like, respectively.

Moreover, if V is space-like and n_x^1 and n_x^2 are the unit normal vectors of C_1 and C_2, respectively, at $x \in C_1 \cap C_2$, then $\langle n_x^1, n_x^2 \rangle = \langle\langle v_1, v_2 \rangle\rangle$. In particular, C_1 and C_2 intersect orthogonally iff $\langle\langle v_1, v_2 \rangle\rangle = 0$.

Two circles $C_i = \mathbb{E}^2 \cap \{v_i\}^\perp$, $1 \leq i \leq 2$, are tangent at a point $\zeta \in \mathbb{E}^2$ if and only if there exists $\lambda \in \mathbb{R}$ such that $v_2 = \pm v_1 + \lambda\zeta$. For $C_i = \mathbb{E}^2 \cap \{v_i\}^\perp$, $1 \leq i \leq 2$, are tangent at $\zeta \in \mathbb{E}^2$ if and only if every circle $C = \mathbb{E}^2 \cap \{v\}^\perp$ that passes trough ζ and is orthogonal to C_1 is also orthogonal to C_2. Hence any unit space-like vector $v \in \{\zeta, v_1\}^\perp$ must also belong to $\{v_2\}^\perp$, and thus $v_2 \in \text{span}\{\zeta, v_1\}$.

Similarly, two straight lines $C_i = \mathbb{E}^2 \cap \{v_i\}^\perp$, $\langle v_i, w \rangle = 0$ for $1 \leq i \leq 2$, are parallel if and only if there exists $\lambda \in \mathbb{R}$ such that $v_2 = \pm v_1 + \lambda w$. For C_1 and C_2 are parallel if and only if every straight line $C = \mathbb{E}^2 \cap \{v\}^\perp$ that is ortogonal to C_1 is also orthogonal to C_2. Hence any unit space-like vector $v \in \{w, v_1\}^\perp$ must also belong to $\{v_2\}^\perp$, and thus $v_2 \in \text{span}\{w, v_1\}$.

4.2 Proof of Proposition 3

Let $\mathcal{F}_i = (S_i^\lambda)_{\lambda \in \Lambda}$, $1 \leq i \leq 2$, be families of straight lines and circles as in Proposition 3. Write $S_i^\lambda = \mathbb{E}^2 \cap \{v_i^\lambda\}^\perp$ for $S_i^\lambda \in \mathcal{F}_i$ and unit space-like vectors v_i^λ, $1 \leq i \leq 2$. Let $V_i \subset \mathbb{L}^4$ be the subspace spanned by the vectors v_i^λ, $1 \leq i \leq 2$. Then the assumption on $\mathcal{F}_1$ and $\mathcal{F}_2$ amounts to saying that $V_1 \subset V_2^\perp$. On the other hand, the fact that $\mathcal{F}_i$ has more than one element implies that the dimension of V_i is at least two. Then either there exists a light-like line L such that $V_1 \cap V_1^\perp = L = V_2 \cap V_2^\perp$, or one of V_1 or V_2, say, V_1, is a space-like plane and V_2 is its (time-like) orthogonal complement.

Assume first that $V_1 \cap V_1^\perp = L = V_2 \cap V_2^\perp$ for the light-like line L spanned by w. Choose any $\zeta \in \mathbb{E}^2$ and a unit space-like vector $v_i \in V_i \cap \zeta^\perp$, $i = 1, 2$. Then $\langle v_1, v_2 \rangle = 0$ and any unit space-like vector $v \in V_i$ can be written as $v = \pm v_i + \lambda w$ for some $\lambda \in \mathbb{R}$. Therefore $C_1 = \mathbb{E}^2 \cap v_1^\perp$ and $C_2 = \mathbb{E}^2 \cap v_2^\perp$ are orthogonal straight lines through ζ and

$\mathcal{F}_i$ is a family of straight lines that are parallel to C_i, $i = 1, 2$.

Suppose now that L is some distinct light-like line spanned by $\zeta \in \mathbb{E}^2$. Choose a unit space-like vector $v_i \in V_i \cap w^\perp$ $i = 1, 2$. Then $\langle v_1, v_2 \rangle = 0$ and any unit space-like vector $v \in V_i$ can be written as $v = \pm v_i + \lambda\zeta$ for some $\lambda \in \mathbb{R}$. Therefore $C_1 = \mathbb{E}^2 \cap v_1^\perp$ and $C_2 = \mathbb{E}^2 \cap v_2^\perp$ are orthogonal straight lines through ζ and $\mathcal{F}_i$ is a family of circles that are tangent to C_i at ζ, $i = 1, 2$. Inverting with respect to a circle centered at ζ transforms $\mathcal{F}_1$ and $\mathcal{F}_2$ into families of straight lines as in the previous case.

Finally, suppose that V_1 is a space-like plane and V_2 is its (time-like) orthogonal complement. If $w \in V_2$, then $\mathbb{E}^2 \cap V_2 = \mathbb{E}^2 \cap V_1^\perp$ consists of a single point $\zeta \in \mathbb{E}^2$, and any $v \in V_1$ is orthogonal to both w and ζ. Therefore $\mathcal{F}_1$ is a family of straight lines through ζ and $\mathcal{F}_2$ is a family of circles centered at ζ.

If $w \notin V_2$, then $\mathbb{E}^2 \cap V_2 = \mathbb{E}^2 \cap V_1^\perp$ consists now of two points $\zeta_1, \zeta_2 \in \mathbb{E}^2$, and any $v \in V_1$ is orthogonal to both ζ_1 and ζ_2. Thus, $\mathcal{F}_1$ is now a family of circles through ζ_1 and ζ_2. Inverting with respect to a circle centered at, say, ζ_1, transforms $\mathcal{F}_1$ into a family of straight lines through the inverse of ζ_2 and $\mathcal{F}_2$ into a family of circles centered at that point. ∎

References

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